BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

DAVID RAYMOND CURTISS LAWRENCE MURRAY GRAVES

PAUL ALTHAUS SMITH

WITH THE ASSISTANCE OF

H. T. DAVIS TOMLINSON FORT G. A HEDLUND

T. H. HILDEBRANDT J. R. KLINE H. W. KUHN

PART I

VOLUME 44

JANUARY TO DECEMBER, 1938

PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island

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CONTENTS

Vol. 44, No. 1 JANUARY, 1938 Whole No. 460

PART 1

Page

Portrait of E. R. HEDRICK ..... Frontispiece

Editorial Preface 1

The October Meeting in New York. By T. R. HOLLCROFT 3

The November Meeting in Iowa City. By M. H. INGRAHAM 7

The November Meeting in Pasadena. By T. M. PUTNAM 9

Cartan on Projectively Connected Spaces. By J. L. VANDERSLICE 11

Remes on Approximation. By J. A. SHOHAT 14

Shorter Notices 16

Notes 24

Abstracts of Papers 28

New Publications 54

CONTENTS

Vol. 44, No. 2 FEBRUARY, 1938 Whole No. 461

Page

Concerning sets of polynomials orthogonal simultaneously on several circles. By G. M. MERRIMAN 57

The resolvent of a closed transformation. By A. E. TAYLOR .. 70

Linear operations on functions of bounded variation. By T. H. HILDEBRANDT 75

A note on linear topological spaces. By D. H. HYERS 76

Note on deduced probability distributions. By R. VON MISES 81

An application of Schläfli's modular equation to a conjecture of Ramanujan. By D. H. LEHMER 84

Extensions of functionals on complex linear spaces. By H. F. BOHNENBLUST and A. SOBCZYK 91

On continued fractions representing constants. By H. S. WALL 94

Existence theorems for solutions of differential equations of non-integral order. By EVERETT PITCHER and W. E. SEWELL 100

Note on integrability conditions of implicit differential equations. By CLYDE M. CRAMLET 107

Differential invariant theory of alternating tensors. By CLYDE M. CRAMLET 110

Note on convex regions on the sphere. By R. M. ROBINSON ... 115

On certain equations in matrices whose elements belong to a division algebra. By M. H. INGRAHAM 117

On the operational determination of two dimensional Green's functions in the theory of heat conduction. By A. N. LOWAN 125

Index numbers and the general economic equilibrium. By F. W. DRESCH 134

Note on almost-universal forms. By P. R. HALMOS 141

Metric spaces with geodesic Ricci curves. I. By JACK LEVINE .. 145

CONTENTS

Vol. 44, No. 3 MARCH, 1938 Whole No. 462

Page

The Annual Meeting of the Society. By T. R. HOLLCROFT.... 153

Frank Morley—In Memoriam. By A. B. COBLE..... 167

Carnap on Logical Syntax. By SAUNDERS MACLANE..... 171

Shorter Notices..... 177

Notes..... 180

Abstracts of Papers..... 185

New Publications..... 205

CONTENTS

Vol. 44, No. 4 APRIL, 1938 Whole No. 463

Page

Singular point problems in the theory of linear differential equations. By W. J. TRJITZINSKY 209

The constructive second number class. By ALONZO CHURCH . . 224

A remark on representations of groups. By TADASI NAKAYAMA . 233

Some properties of functions of exponential type. By RICHARD DUFFIN and A. C. SCHAEFFER 236

Approximation to a given function by means of polynomials in another given function. By W. C. RISSELMAN 241

A quadratic form problem in the calculus of variations. By A. A. ALBERT 250

The Riemannian curvature of a hypersurface. By AARON FIALKOW 253

Matric conjugates in a ring $R(A)$. By H. S. THURSTON 258

Abstract residuation over lattices. By R. P. DILWORTH 262

A note on the maximum principle for elliptic differential equations. By FRITZ JOHN 268

Birational transformations in 4-space and 5-space. By A. R. WILLIAMS 272

On completely continuous linear transformations. By INGO MADDAUS 279

Multivalent functions of order p. By M. S. ROBERTSON 282

On the order of the partial sums of a Fourier series. By W. C. RANDELS 286

On some inequalities of S. Bernstein and W. Markoff for derivatives of polynomials. By A. C. SCHAEFFER and R. J. DUFFIN . 289

Some theorems on subsequences. By H. J. HAMILTON 298

CONTENTS

Vol. 44, No. 5 **MAY, 1938** **Whole No. 464**

Page

The February Meeting in New York. By T. R. HOLLCROFT... 305

Carathéodory on Geometrical Optics. By J. P. C. SOUTHALL.. 309

Volterra and Pérès on Functionals. By M. R. HESTENES..... 311

Shorter Notices..... 313

Notes..... 322

Abstracts of Papers..... 328

New Publications..... 357

CONTENTS

Vol. 44, No. 6 JUNE, 1938 Whole No. 465

Page

The present status of the theory of electrolytes. By C. A. KRAUS 361

Torsion of regions bounded by circular arcs. By I. S. SOKOLNIKOFF and E. S. SOKOLNIKOFF 384

A multiplier rule in abstract spaces. By H. H. GOLDSTINE 388

On the nth derivative of $f(x)$. By H. S. WALL 395

The geometry of the whirl-motion group G_6: elementary invariants. By EDWARD KASNER and JOHN DE CICCO 399

A note on numbers of the form $a^2+\alpha b^2+\beta c^2+\alpha\beta d^2$. By R. M. THRALL 404

A moment-generating function which is useful in solving certain matching problems. By E. G. OLDS 407

A theorem on interior transformations. By G. T. WHYBURN .. 414

On the transformation group for diabolic magic squares of order four. By BARKLEY ROSSER and R. J. WALKER 416

A note on regular Banach spaces. By B. J. PETTIS 420

Relations among the fundamental solutions of the generalized hypergeometric equation when $p=q+1$: I. Non-logarithmic cases. By F. C. SMITH 429

A note on Fredholm-Stieltjes integral equations. By F. G. DRESSEL 434

A theorem on quadratic forms. By W. T. REID 437

On the extended form of Cauchy's condensation test for the convergence of infinite series. By R. E. MORITZ 441

On the boundary condition $\partial u/\partial n+au=0$ for harmonic functions. By HILLEL PORITSKY 443

On generating the simple group $LF(2, 2^N)$ by two operators of periods two and three. By ABRAHAM SINKOV 449

Note on the defining relations for the simple group of order 660. By F. A. LEWIS 456

CONTENTS

Vol. 44, No. 7 JULY, 1938 Whole No. 466

Page

The April Meeting in Chicago. By W. L. AYRES 457

The April Meeting in Berkeley. By T. M. PUTNAM 461

The April Meeting in Charlottesville. By T. R. HOLLCROFT .. 463

Albert on Modern Higher Algebra. By H. W. BRINKMANN 471

Hilbert and Ackermann on Logic. By BARKLEY ROSSER 474

Doetsch on the Laplace Transformation. By R. V. CHURCHILL. 476

Shorter Notices 478

Notes 480

Abstracts of Papers 484

New Publications 494

CONTENTS

Vol. 44, No. 8 AUGUST, 1938 Whole No. 467

Page

The topology of transformation groups. By P. A. SMITH..... 497

Note on an elementary problem of interpolation. By P. ERDÖS and G. GRÜNWALD..... 515

On the Peano curve of Lebesgue. By I. J. SCHOENBERG..... 519

Note on the curvature of orthogonal trajectories of level curves of Green's functions. By J. L. WALSH..... 520

A note on linear functionals. By R. P. BOAS, JR., and J. W. TUKEY 523

On certain groups of birational contact transformations. By J. M. FELD..... 529

Polygonal variations. By ALINE H. FRINK and ORRIN FRINK.. 539

A property of harmonic functions in three variables. By J. W. GREEN..... 548

On the class of metrics defining a metrisable space. By H. E. VAUGHAN..... 557

Formal synthesis of two periodic correspondences, of period five and seven, respectively. By H. S. WHITE..... 562

Divisibility of generalized factorials. By BENJAMIN ROSENBAUM..... 566

Remarks on the classical inversion formula for the Laplace integral. By D. V. WIDDER and N. WIENER..... 573

Non-cyclic algebras with pure maximal subfields. By A. A. ALBERT..... 576

The transforms of Fuchsian groups. By P. K. REES..... 580

A generalization of a property of harmonic functions. By MAX CORAL..... 587

CONTENTS

Vol. 44, No. 9 SEPTEMBER, 1938 Whole No. 468

Page

The International Congress of Mathematicians. By the ORGANIZING COMMITTEE 593

Harry Walter Tyler—In Memoriam. By H. B. PHILLIPS 596

Cartan on Groups and Differential Geometry. By HERMANN WEYL 598

Courant and Hilbert on Partial Differential Equations. By HERMANN WEYL 602

Contributions to the Calculus of Variations. By ARNOLD DRESDEN 604

Shorter Notices 610

Notes 617

Abstracts of Papers 623

New Publications 645

CONTENTS

Vol. 44, No. 10 OCTOBER, 1938 Whole No. 469

Page

A note on normal division algebras of prime degree. By A. A. ALBERT 649

Quelques remarques sur les relations entre les notions d'écart régulier et de distance. By N. ARONSZAJN 653

A symmetric representation of the twenty-seven lines on a cubic surface by lines in a finite geometry. By J. S. FRAME 658

A sufficiency proof for isoperimetric problems in the calculus of variations. By M. R. HESTENES 662

A new summation method for divergent series. By W. A. MERSMAN 667

Metric properties of the cylinder of Kubota. By RUTH B. RASMUSEN 674

Apolarity of trilinear forms and pencils of bilinear forms. By R. M. THRALL 678

On some new congruences in the theory of Bernoulli's numbers. By N. G. W. H. BEEGER 684

Concerning R. L. Moore's Axiom 5. By F. B. JONES 689

Enumerative properties of plane connected n-lines. By B. C. WONG 693

A relative of the lemma of Schwarz. By E. F. BECKENBACH .. 698

Some iterated integrals in the fractional calculus. By M. S. MACPHAIL 707

Transformation of bases for relative linear sets. By MARGARETE C. WOLF 716

A note on the elementary divisor theory in non-commutative domains. By TADASI NAKAYAMA 719

On a mixed boundary condition for harmonic functions. By HILLEL PORITSKY 723

Polygenic functions whose associated element-to-point transformation converts unions into points. By EDWARD KASNER . 726

On the absolute summability of Fourier series. By W. C. RANDELS 733

Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication. By TANG TSAO-CHEN .. 737

CONTENTS

Vol. 44, No. 11 NOVEMBER, 1938 Whole No. 470

Page

The Summer Meeting in New York. By T. R. HOLLCROFT... 745

The Chicago Conference and Seminar on Algebra. By A. A. ALBERT..... 756

Shorter Notices..... 758

Notes..... 768

Abstracts of Papers..... 772

New Publications..... 791

CONTENTS

Vol. 44, No. 12 DECEMBER, 1938 Whole No. 471

PART I

Page

Lattices and their applications. By GARRETT BIRKHOFF..... 793

On the application of structure theory to groups. By OYSTEIN ORE..... 801

The representation of Boolean algebras. By M. H. STONE.... 807

The applicability of lattice theory to group theory. By REINHOLD BAER..... 817

Non-euclidean geometry of joining and intersecting. By KARL MENGER..... 821

The application of lattice theory to integration. By H. M. MACNEILLE..... 825

On fundamental functions of Lagrangean interpolation. By PAUL ERDÖS and B. A. LENGYEL..... 828

Completely continuous transformations in Hilbert space. By F. SMITHIES..... 835

Un problème de la théorie des nombres rattaché aux polynômes de Tschebycheff. By ERVIN FELDHEIM..... 836

The equivalence of sequence integrals and non-absolutely convergent integrals. By R. L. JEFFERY..... 840

The relation of perfect sets of measure zero to certain classes of functions. By P. T. MAKER..... 846

On Fourier series with restricted coefficients. By OTTO SZÁSZ.. 850

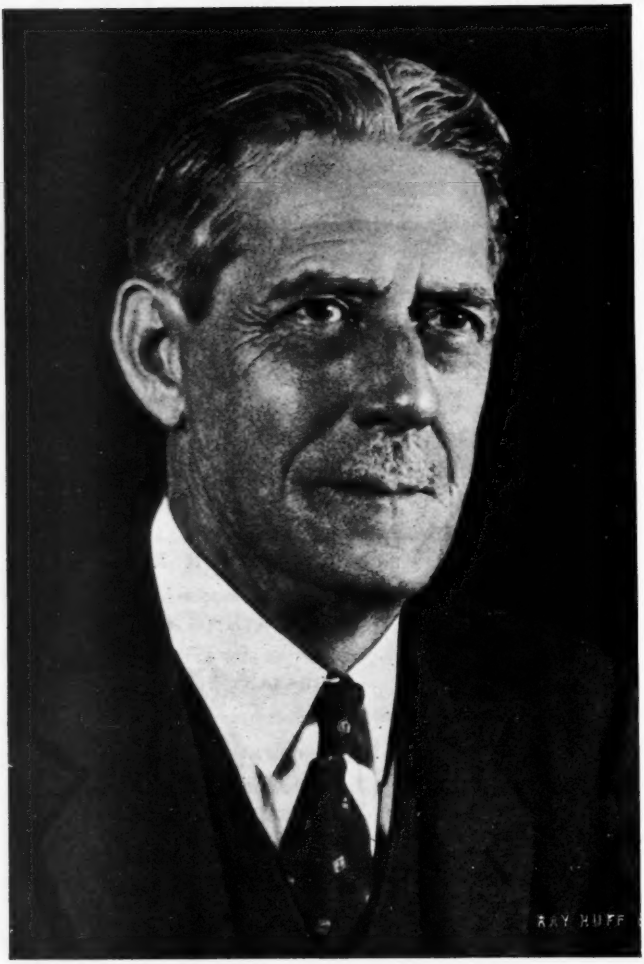
Spin representation of inversions. By A. H. TAUB..... 860

Note on degree of trigonometric and polynomial approximation to an analytic function. By J. L. WALSH and W. E. SEWELL..... 865

Expansion of functions in solutions of functional equations. By LEONARD BRISTOW..... 874

Orthogonal polynomials with orthogonal derivatives. By M. S. WEBSTER..... 880

A correction. By EVERETT PITCHER and W. E. SEWELL..... 888


RAY HUFF

By order of the Council of the American Mathematical Society, this volume is dedicated to

EARLE RAYMOND HEDRICK

in appreciation of his services to the Society as Editor-in-Chief of this Bulletin for the past seventeen years.

BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

Editorial Preface

On March 19, 1937, Professor E. R. Hedrick took office as Vice President of the University of California and Provost of the University of California at Los Angeles, and in consequence found it necessary to resign as editor-in-chief of this Bulletin, though continuing on the editorial board until the end of the year. In recognition of his extraordinary services to this journal, the present volume is dedicated to him. The Council of the Society also adopted unanimously the following resolution of appreciation, at its March meeting:

Whereas, Professor Earle Raymond Hedrick on March 19, 1937, took office as Vice President and Provost of the University of California at Los Angeles and will find it necessary at the end of 1937 to relinquish all connection with our Bulletin of which he has been an able editor-in-chief for seventeen years; and

Whereas, for more than three decades, Professor Hedrick has given abundantly of his time, thought, and effort in the service of the Society, as an elected member of the Council (1905–1907), as Vice President (1916), as Editor (1921–1937), as President and Trustee (1929–1930), and as a member of many important committees; and

Whereas, in all of his duties as an officer of our Society, and in many other ways both within and without the Society, Professor Hedrick has labored unceasingly and with conspicuous success to advance the interests and prestige of mathematics and mathematicians, including teachers in all grades and research workers in a variety of fields; and

Whereas, in particular, his services as editor-in-chief of the Bulletin for seventeen years, involving an expenditure of time and energy which only those associated with him in this work can fully appreciate, have greatly contributed to the success of this publication and its usefulness to the members of this Society;

Therefore: Be it Resolved: that the Council of the American Mathematical Society hereby tenders to Professor Hedrick its profound appreciation of his many services to the cause of mathematics,

and its very sincere wishes for success in the great work upon which he has now entered, and instructs the Secretary to send to him a copy of this Resolution.

The retirement of Professor Hedrick is but one of several changes in the editorial staff of the BULLETIN. Professor W. R. Longley also has asked to be relieved, after twelve years of service. His place and that of Professor Hedrick have been filled by the appointment of Professors L. M. Graves and P. A. Smith. During part of the year the latter had served as assistant editor, replacing D. N. Lehmer. He is now succeeded by Professor Tomlinson Fort, who will be in charge of book reviews, formerly handled by Professor Longley. Professor H. L. Rietz has also retired after having served as assistant editor since 1925, and Professor G. A. Hedlund has taken his place.

In accordance with the advice of the Council, the new board of editors is endeavoring to divide editorial work and responsibility as evenly as possible. For the present year Professors Graves and Smith will take charge of contributed papers; manuscripts in Analysis and Algebra should be sent to the former, those in other fields to the latter. Professor Curtiss will continue as managing editor, and will be in charge of the odd-numbered issues, which are devoted to reports of meetings, reviews, notes, abstracts, and new publications.

No major changes as regards either the policy or the contents of the BULLETIN are contemplated. Although extraordinary circumstances have led to the acceptance recently of three papers (not invited addresses) running slightly over the ten page limit which has been in force for several years, it is intended to adhere to the former rule after the next (February) issue. The reader will, however, note a number of minor typographical changes. A few of these have been made with the hope of improving the appearance both of covers and of pages. Others will tend to make styles for the TRANSACTIONS and the BULLETIN more uniform. Another innovation, perhaps less obvious, is the increase in amount of printed matter per page, made possible by smaller margins. This will reduce the number of pages for each volume by about ten per cent.

A large part of the burden of proof-reading formerly borne by editors and referees will now be assumed by the New York office. Certain minor changes in procedure with proofs will result.

We hope that the readers of the BULLETIN will agree with us as to the desirability of these new styles and procedures, and will sympathize with us in our efforts to continue the improvement of this journal so notable under the administration of Professor Hedrick.

THE EDITORS

THE OCTOBER MEETING IN NEW YORK

The three hundred forty-fourth meeting of the American Mathematical Society was held at Columbia University on Saturday, October 30, 1937. The attendance included the following one hundred sixty-six members of the Society:

H. S. Adams, R. P. Agnew, E. B. Allen, R. L. Anderson, R. G. Archibald, F. E. Baker, N. H. Ball, M. F. Becker, D. L. Bernstein, W. J. Berry, G. D. Birkhoff, E. M. Blake, R. P. Boas, H. W. Bode, H. F. Bohnenblust, Samuel Borofsky, Joseph Bowden, C. B. Boyer, A. D. Bradley, A. B. Brown, B. L. Brown, R. S. Burington, J. H. Bushey, J. Hughes Bushey, W. E. Byrne, J. W. Calkin, R. H. Cameron, A. D. Campbell, Alonzo Church, G. A. Coon, T. F. Cope, H. B. Curry, J. H. Curtiss, Tobias Dantzig, Norman Davidowitz, J. J. DeCicco, E. L. Dodd, Arnold Dresden, Melvin Dresher, D. M. Dribin, H. T. Engstrom, J. M. Feld, Aaron Fialkow, T. S. Fiske, W. B. Fite, W. W. Flexner, R. M. Foster, J. S. Frame, Sr. Francis Xavier, Philip Franklin, Kurt Friedrichs, D. L. Fuller, J. J. Gergen, B. P. Gill, J. W. Givens, Saul Gorn, S. L. Greitzer, C. C. Grove, Margaret Gurney, Marshall Hall, Philip Hartman, G. G. Harvey, Robert Henderson, E. H. C. Hildebrandt, Einar Hille, Banesh Hoffmann, Lulu Hofmann, T. R. Hollcroft, M. A. Holly, E. M. Hull, Witold Hurewicz, W. H. Ingram, S. A. Joffe, M. M. Johnson, H. A. Jordan, I. N. Kagno, E. R. van Kampen, L. S. Kennison, J. R. Kline, M. S. Knebelman, L. A. Knowler, B. O. Koopman, Jack Laderman, Solomon Lefschetz, C. A. Lester, Madeline Levin, Harry Levy, Marie Litzinger, E. R. Lorch, L. A. Lorch, A. N. Lowan, N. H. McCoy, J. C. C. McKinsey, L. A. MacColl, C. C. MacDuffee, H. M. MacNeille, R. J. Marcou, A. J. Maria, Sr. Marie McNeil, Walther Mayer, Karl Menger, F. H. Miller, A. K. Mitchell, E. C. Molina, Deane Montgomery, C. B. Morrey, Richard Morris, A. P. Morse, Marston Morse, G. W. Mullins, F. J. Murray, D. S. Nathan, C. A. Nelson, Rufus Oldenburger, Oystein Ore, E. K. Paxton, W. F. Penney, E. L. Post, Walter Prenowitz, J. F. Randolph, H. W. Raudenbush, M. S. Rees, Moses Richardson, R. G. D. Richardson, J. F. Ritt, J. H. Roberts, M. S. Robertson, S. L. Robinson, J. B. Rosser, O. F. G. Schilling, Hyman Serbin, Stephan Serghiesco, L. P. Siceloff, James Singer, M. F. Smiley, P. A. Smith, W. C. Strodt, D. J. Struik, R. R. Struik, M. M. Sullivan, J. D. Tamarkin, A. E. Taylor, J. M. Thomas, E. W. Titt, C. B. Tompkins, J. W. Tukey, Annita Tuller, J. L. Vanderslice, Oswald Veblen, R. J. Walker, H. S. Wall, Louis Weisner, M. J. Weiss, M. E. Wells, Hermann Weyl, A. P. Wheeler, A. L. Whiteman, J. K. Whittemore, D. V. Widder, Norbert Wiener, S. S. Wilks, Aurel Wintner, H. P. Wirth, Leo Zippin, O. J. Zobel, H. S. Zuckerman.

Two sections, one for Algebra and Topology, the other for Analysis and Differential Geometry, met on Saturday morning. At a general session Saturday afternoon, Professor D. J. Struik, of the Massachusetts Institute of Technology, gave an address entitled *The application of tensor analysis to problems of electrical engineering*.

The general session opened with a short business meeting at which the members of the Society voted, on recommendation of the Coun-

cil, to amend the phrase "three Associate Secretaries" in Article I, Section 1 of the By-Laws to read "four Associate Secretaries."

The Board of Trustees met at 6 P.M. Friday, October 29, in the Faculty Club of Columbia University. A tentative budget for 1938 was adopted. Mr. S. A. Joffe and Professor A. E. Meder, Jr. were appointed auditors of the accounts for the current fiscal year. Professor M. H. Ingraham was elected Secretary of the Board of Trustees for a term ending December 31, 1938. The prices of the Colloquium Publications were revised. The Charles Phelps Taft Memorial Fund of the University of Cincinnati was thanked for a generous subvention of $600 toward the publication of the forthcoming Colloquium volume, *Summable Series and Convergence Factors*, by Professor C. N. Moore.

A meeting of the Council was held at 1:00 P.M. Saturday, October 30, in the Faculty Club.

Three persons were elected to membership in the Society:

Mrs. Mary Piper Hollcroft (Mrs. T. R.), Wells College;
Mr. Wilfred Kaplan, Jamaica Plain, Mass.;
Dr. James Keene Lorne MacDonald, Cornell University.

Mr. Philip Hall, King's College, Cambridge, England, was admitted to the Society under the reciprocity agreement with the London Mathematical Society.

The following appointments by President R. L. Moore were reported: as committee to nominate representatives on the National Research Council and the Council of the American Association for the Advancement of Science: Professors J. L. Walsh (chairman), C. G. Latimer, and T. Y. Thomas; as committee on fees for Summer Meetings: Professors M. H. Ingraham (chairman), P. A. Smith, and J. D. Tamarkin.

The Council voted to dedicate the 1938 volume of the Bulletin to Earle Raymond Hedrick in appreciation of his services to the Society as Editor-in-Chief of the Bulletin for the past seventeen years.

The Council, with the approval of the Board of Trustees, appointed Professor W. L. Ayres, of the University of Michigan, Associate Secretary for the remainder of the year 1937, to fill the vacancy created by vote of the Society at this meeting.

It was announced that Professors A. A. Albert and M. H. Stone had accepted invitations to deliver each a series of Colloquium Lectures in 1939.

Titles and cross references to abstracts of papers presented at this meeting are given below. Papers numbered 1 to 8 were read before

the section for Algebra and Topology, Professor C. C. MacDuffee presiding; papers numbered 9 to 16 were read before the section for Analysis and Differential Geometry, Professor Karl Menger presiding. Papers numbered 17 to 24, whose abstract numbers are followed by the letter *t*, were read by title. Vice President Norbert Wiener presided at the afternoon session. Professor Tadasi Nakayama was introduced by Professor Hermann Weyl and Professor A. E. Staniland by Professor J. S. Taylor. Paper number 2 was read by Dr. Nesbitt, 6 by Dr. Zippin, and 13 by Dr. Boas.

1. *A determination of the automorphisms of certain algebraic fields*, by Caroline A. Lester. (Abstract 43-9-370.)

2. *On modular representations of groups of finite order*, by Richard Brauer and Cecil Nesbitt. (Abstract 43-11-374.)

3. *A remark on representations of groups*, by Tadasi Nakayama. (Abstract 43-11-382.)

4. *Note on the four color problem*, by Philip Franklin. (Abstract 43-11-378.)

5. *Some properties of the general or substitutive mean*, by E. L. Dodd. (Abstract 43-9-368.)

6. *On abelian groups of auto-homeomorphisms of metric spaces*, by Deane Montgomery and Leo Zippin. (Abstract 43-11-381.)

7. *Decomposition of forms*, by Rufus Oldenburger. (Abstract 43-11-383.)

8. *A problem connected with the "axiomatic of affine geometry,"* by Ruth R. Struik. (Abstract 43-11-426.)

9. *Fields whose geodesic series have circles for their point- or line-unions*, by J. J. DeCicco. (Abstract 43-11-377.)

10. *Hypersurfaces of a space of constant curvature*, by Aaron Fialkow. (Abstract 43-9-369.)

11. *Analytic transformations in euclidean space of 2n dimensions*, by A. E. Staniland. (Abstract 43-9-359.)

12. *Linear functionals satisfying prescribed conditions*, by R. P. Agnew. (Abstract 43-11-372.)

13. *The iterated Stieltjes transform*, by R. P. Boas, Jr. (National Research Fellow) and D. V. Widder. (Abstract 43-11-373.)

14. *The distribution of values of an analytic almost periodic function in equally spaced circles*, by R. H. Cameron. (Abstract 43-11-375.)

15. *Uniform convergence and summability of the Jacobi series on an unrestricted lemniscate*, by J. H. Curtiss. (Abstract 43-11-376.)

16. *Mean motion of almost periodic functions and the Riemann zeta function*, by Philip Hartman (Abstract 43-11-424.)

17. *On the numerical solution of integral equations* (preliminary report), by W. H. Ingram. (Abstract 43-7-285-*t*.)

18. *On fourth order self-adjoint difference systems*, by V. V. Latshaw. (Abstract 43-11-397-*t*.)

19. *Note on the operational determination of Green's functions in the theory of heat conduction*, by A. N. Lowan. (Abstract 43-11-380-*t*.)

20. *The quaternion congruence* $\overline{\tau}\alpha\tau \equiv \beta \pmod{p}$, by R. E. O'Connor, S.J. (Abstract 43-9-371-*t*.)

21. *A note on regular Banach spaces*, by B. J. Pettis. (Abstract 43-11-384-*t*.)

22. *On a functional equation which admits a lacunary function as a solution*, by L. B. Robinson. (Abstract 43-11-385-*t*.)

23. *Magic squares of the fifth order*, by F. H. Safford. (Abstract 43-7-313-*t*.)

24. *The place of Euclid's tenth book in ancient and modern science*, by Ruth R. Struik. (Abstract 43-11-425-*t*.)

T. R HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN IOWA CITY

The three hundred forty-fifth meeting of the Society was held at the State University of Iowa on Friday and Saturday, November 26–27, 1937. At this meeting there were in attendance about one hundred persons, among whom were the following fifty-six members of the Society:

E. S. Allen, B. M. Armstrong, J. V. Atanasoff, G. A. Bliss, L. M. Blumenthal, H. R. Brahana, W. C. Brenke, C. C. Camp, H. H. Campaigne, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, A. H. Copeland, A. T. Craig, W. M. Davis, H. H. Goldstine, Cornelius Gouwens, L. M. Graves, G. L. Gross, G. A. Herr, R. A. Higdon, J. J. L. Hinrichsen, D. L. Holl, M. H. Ingraham, Dunham Jackson, D. E. Kearney, I. F. Keeler, R. B. Kershner, S. C. Kleene, E. H. Larguier, M. I. Logsdon, J. V. McKelvey, Dorothy Manning, R. J. Michel, E. E. Moots, J. L. Nagle, E. N. Oberg, G. B. Price, W. C. Randels, W. T. Reid, J. F. Reilly, H. L. Rietz, P. G. Robinson, W. J. Rusk, R. G. Sanger, E. R. Smith, R. C. Stephens, E. B. Stouffer, W. J. Trjitzinsky, Henry Van Engen, R. W. Wagner, L. E. Ward, W. G. Warnock, K. W. Wegner, K. P. Williams, Roscoe Woods.

The meeting opened Friday afternoon with a session for contributed papers followed by an address by Professor W. J. Trjitzinsky, of the University of Illinois, entitled *Singular point problems in the theory of linear differential equations*. Saturday morning Professor A. H. Copeland, of the University of Michigan, spoke on *Fundamental concepts of the theory of probability*, and the meeting closed with another session for short papers.

One of the features of the meeting was a session in the old Iowa State Capitol, a building of marked architectural distinction.

On Friday evening a dinner was held at the Memorial Union. About seventy-five mathematicians and their guests attended. Professor H. L. Rietz acted as toastmaster and called first upon Professor Dunham Jackson. Dean E. B. Stouffer then spoke on the problems presented to mathematics by the reorganization of the college curriculum which would be necessary if the present first two years of college work were turned over to the high schools. Professor G. A. Bliss expressed the feeling that the meetings which are most enjoyable and of greatest value to the members attending them are the smaller meetings, such as the one at Iowa City, at which one can attend the entire scientific program and at which there is ample time for the discussion of each paper.

Professor E. W. Chittenden presided at the general session on Friday, and Professor W. C. Brenke at the lecture. Professor E. R.

Smith was chairman at the lecture on Saturday morning, and Professor L. E. Ward at the session which followed. At the end of the first session on Friday afternoon Professor J. F. Reilly read a statement concerning Professor R. P. Baker, of the University of Iowa, who recently died.

The titles of papers read at the meeting follow. Those whose abstract numbers are followed by *t* were read by title. Mr. J. H. Daoust was introduced by Professor Dunham Jackson, Professor Gerhard Tintner by Professor E. R. Smith, Mr. C. E. Sealander by Professor L. E. Ward, Mr. Fulton Koehler by Professor Dunham Jackson, Mr. F. D. Rigby by Professor E. W. Chittenden, and Dr. E. H. Rothe by Professor M. H. Ingraham. Papers 1–8 were presented Friday afternoon and papers 9–17 on Saturday morning.

1. *On the weak summability of Fourier series*, by W. C. Randels. (Abstract 43-11-393.)

2. *On certain constants associated with Bernstein's theorem*, by J. H. Daoust. (Abstract 43-11-401.)

3. *On a van der Corput absolute constant*, by R. B. Kershner. (Abstract 43-11-403.)

4. *On the correlation of a mean and standard deviation in small samples drawn from a certain non-normal population*, by H. L. Rietz. (Abstract 44-1-2.)

5. *On correlation due to common elements*, by A. T. Craig. (Abstract 44-1-1.)

6. *A dynamic theory of duopoly*, by Gerhard Tintner. (Abstract 44-1-5.)

7. *Some implicit function theorems in the large*, by G. M. Ewing. (Abstract 43-11-398-*t*.)

8. *Topological proofs of uniqueness theorems in the theory of differential and integral equations*, by E. H. Rothe. (Abstract 43-11-423.)

9. *On a family of curves associated with a conjugate net*, by P. O. Bell. (Abstract 43-11-399-*t*.)

10. *A particular third order irregular boundary value problem* (preliminary report), by C. E. Sealander. (Abstract 44-1-4.)

11. *The characterization of pseudo-$S_{2,r}$-sets*, by L. M. Blumenthal. (Abstract 43-11-397.)

12. *Note on orthogonal polynomials in three variables*, by Dunham Jackson. (Abstract 43-11-400.)

13. *Orthogonal polynomials on a square*, by Fulton Koehler. (Abstract 43-11-402.)

14. *Note on the axioms for Boolean algebras*, by F. D. Rigby. (Abstract 44-1-3.)

15. *Additional notes on the inversion of the Laplace transformation*, by R. V. Churchill. (Abstract 43-11-394-*t*.)

16. *A quadratic form problem in the calculus of variations*, by A. A. Albert. (Abstract 43-11-395-*t*.)

17. *Weakly complete Banach spaces*, by H. H. Goldstine. (Abstract 43-11-396-*t*.)

M. H. INGRAHAM,
Associate Secretary

THE NOVEMBER MEETING IN PASADENA

The three hundred forty-sixth meeting of the American Mathematical Society was held at the California Institute of Technology on Saturday, November 27, 1937. The chair was occupied at various times by Professors E. R. Hedrick, G. C. Evans, and E. T. Bell. The attendance of about fifty included the following thirty-one members of the Society:

L. D. Ames, H. A. Arnold, H. M. Bacon, Clifford Bell, E. T. Bell, Myrtie Collier, P. H. Daus, R. P. Dilworth, C. H. Dix, G. C. Evans, H. E. Glazier, J. W. Green, D. W. Hall, N. A. Hall, H. J. Hamilton, E. R. Hedrick, C. G. Jaeger, Glenn James, W. E. Mason, A. D. Michal, T. M. Putnam, W. C. Risselman, A. A. Shaw, D. V. Steed, S. E. Urner, F. A. Valentine, Morgan Ward, M. B. White, W. M. Whyburn, E. R. Worthington, Max Zorn.

Luncheon for members and their guests was served at the Athenaeum of the California Institute of Technology.

The titles of papers read at the meeting follow. Those whose abstract numbers are followed by the letter *t* were read by title.

1. *Some theorems on subsequences*, by H. J. Hamilton. (Abstract 43-11-404.)

2. *Abstract residuation over lattices*, by R. P. Dilworth. (Abstract 43-11-405.)

3. *A formal expansion theory for functions defined by two variable power series*, by N. A. Hall. (Abstract 43-11-406.)

4. *A property of harmonic functions in three variables*, by J. W. Green. (Abstract 43-11-407.)

5. *Properties of invariant sets under pointwise periodic homeomorphisms*, by D. W. Hall and G. E. Schweigert. (Abstract 43-11-408.)

6. *A new fragment of Euclid's "Elements"* (preliminary report), by A. A. Shaw. (Abstract 43-11-409.)

7. *On the first case of Fermat's last theorem*, by Glenn James. (Abstract 43-11-410.)

8. *Iterated exponential numbers*, by E. T. Bell. (Abstract 43-11-411.)

9. *Differentiation in linear, semi-ordered spaces*, by H. A. Arnold. (Abstract 43-11-412.)

10. *General Riemannian spaces with abstract coordinates and constant curvature*, by A. D. Michal. (Abstract 43-11-413.)

11. *On the elliptic quintic curve in space of four dimensions*, by A. H. Diamond. (Abstract 43-11-414.)

12. *Topological studies in the theory of analytic functions* (preliminary report), by Max Zorn. (Abstract 43-11-415.)

13. *Structure residuation*, by Morgan Ward. (Abstract 43-11-416.)

14. *Collineations and central projections*, by P. H. Daus. (Abstract 43-11-417-*t*.)

15. *Note on convex regions on the sphere*, by R. M. Robinson. (Abstract 43-11-418-*t*.)

16. *Lagrange and Wilson theorems for the generalized Stirling numbers*, by E. T. Bell. (Abstract 43-11-419-*t*.)

17. *Notes on denumerants*, by E. T. Bell. (Abstract 43-11-420-*t*.)

18. *Riemannian tensor analysis in abstract spaces*, by H. A. Arnold. (Abstract 43-11-421-*t*.)

19. *Infinitely dimensional Riemannian differential geometries with constant curvature*, by A. D. Michal. (Abstract 43-11-422-*t*.)

T. M. PUTNAM,
Associate Secretary

CARTAN ON PROJECTIVELY CONNECTED SPACES

Leçons sur la Théorie des Espaces à Connexion Projective. By Élie Cartan. Edited by P. Vincensini. (Cahiers Scientifiques, publiés sous la direction de M. Gaston Julia, no. 17.) Paris, Gauthier-Villars, 1937. vi+308 pp.

In a recent monograph, La Méthode de Repère mobile, · · · (Actualités Scientifiques et Industrielles, no. 194; reviewed in this Bulletin, November, 1935) E. Cartan presented a splendid outline of his general method of approach toward all branches of differential geometry. In a word, it consists of a far-reaching generalization of the familiar moving trihedral with assistance from the theory of groups. The analytical formulation employs the exterior differential calculus, a discipline extensively used by Cartan since the turn of the century. And there is frequent recourse to the theory of Pfaffian systems. Undoubtedly his unusual analytical machinery is, to many, a source of difficulty. Most differential geometers use, instead, Ricci's tensor calculus and theorems on total differential equations stemming from Christoffel. But every disciple of Ricci knows the profit which lies in the study of Cartan.

In the above monograph the author devoted one paragraph to showing how projective differential geometry fits into his general scheme. In the work here under review this paragraph is presented to us in more satisfactory form as a book of three hundred odd pages.

The book has two principal divisions, the first devoted to classical projective differential geometry, the second, to the geometry of projectively connected spaces. The first chapter is concerned with the projective line, both real and complex. Since from the intrinsic standpoint any two one-dimensional loci are locally equivalent, the purely differential-geometric discussion of the real line is trivial. But by introducing motion, Cartan finds a kinematical theory which is a good introduction to his method of moving reference systems, systems which in this book are always simplexes defining homogeneous projective coordinates. A pleasing detail is his interpretation of the Schwarzian derivative as a projective acceleration. The material on the complex line shows how moving reference systems can be used in the complex domain. This subject he has treated at length in a previous work.

In the second chapter the elements of the theory of plane curves is first developed by Wilczynski's method. Here the author adds an exciting definition of the projective arc. Then, returning to his own method of moving reference systems, he redevelops the theory twice and in some detail, first with aid from the method of reduced equations, and finally directly. In the direct method geometrical intuition is ignored and the invariant theory is developed purely analytically. This attack makes for difficulty but has the advantage of being applicable in general situations where intuition fails to suggest shortcuts. In the present case it leads to an intrinsic reference simplex of the sixth order associated with each point of a curve, in terms of which the projective analogues of the Frenet equations take their simplest form. The chapter ends with a derivation of the so-called structure equations of the projective group and a simplification of the direct method by their use. At one point in this chapter the author makes the surprising statement that curves of constant projective curvature (for example, $y=x^3$) have no inflection points.

The third chapter carries the first division of the work to its conclusion with a discussion of surfaces in three-space. The method of reduced equations leads to the

differential forms of Fubini and the projective line element. The quadrics and lines of Darboux and the quadric of Lie receive an unusually clear analytic and geometric characterization. Strangely enough the usual interpretation of the linear element as a cross ratio is not mentioned. The author then proceeds to apply his direct method of moving reference systems. Starting with general coordinate simplexes associated with each point of the surface, restrictions of successively higher differential order are applied until a unique system of the fourth order is invariantively determined at each point. This development leads to the six fundamental invariants of a surface. The chapter ends with an interesting section on projectively applicable surfaces, in particular, on the order of contact obtainable at homologous points.

The second division of the work opens by crossing the divide between projective space and a space with a projective connection. Since most investigators seem content to stay on one side or the other, projective differential geometry has been strongly divided into two schools. The author, however, belongs to both, and with fine skill carries us from the one to the other without unpleasant jolts.

The projective connection of an n-space with general coordinates u^i consists of $n(n+2)$ Pfaffians $\omega_\beta^\alpha = \prod_{\beta i}^{\alpha}(u)du^i$, with $\omega_0^0=0$ and $\alpha,\beta=0, 1, \cdots, n$, which define a transformation $dA_\alpha=\omega_\alpha^\beta A_\beta$ from a local reference simplex, of vertices A_α, at u^i to a simplex, of vertices $A_\alpha+dA_\alpha$, at u^i+du^i. The simplex at each point u^i is regarded as lying in a projective n-space associated with this point, the points A_0 and u^i being identified. It is shown that the connection determines an invariantive application of the second order between a neighborhood of any point u^i of the base space and a neighborhood of A_0 in the space associated with u^i. Abstractly, therefore, the associated spaces may be regarded as tangent spaces. When the torsion is zero the above application may be made to the third order. Thus Cartan uses the assigned connection to batten down the associated spaces. A slightly more general procedure followed by some writers is to batten down the associated spaces before assigning the connection. The author considers how the ω_β^α change when the local reference simplexes are changed, and arrives at a certain "natural" reference system to be associated with a given system in the base space by normalizing the ω_β^α in an invariant way.

In Chapter 2 the tensor of curvature and torsion is derived in the usual manner by displacing a tangent space about an infinitesimal parallelogram in the base space. The equations obtained by setting this tensor equal to zero are just the equations of structure of the projective group. Hence the vanishing of this tensor is necessary and sufficient for the generalized space to be locally projective. This leads in the following chapter to a discussion of projective tensor calculus and there results a most interesting and unusual development of this discipline. It is an application of the general theory presented by the author in his address to the Society at the Harvard Tercentenary, published in the Annals of Mathematics (January, 1937). His projective tensors and their algebra turn out to be essentially the same as Veblen's. Different, however, is his covariant differentiation, defined geometrically and applicable to any field quantity. But even when applied to a tensor it does not give rise in general to another tensor. As an application of this calculus the author proves that a space with a projective connection is completely characterized by the curvature-torsion tensor and its successive covariant derivatives. The proof rests on the use of normal coordinates (usual definition) and particular reference simplexes invariantively associated with them. At the origin of normal coordinates Cartan's covariant derivatives reduce to ordinary derivatives.

After a short chapter devoted to the projective generalization of the identities of

Bianchi and their geometrical interpretation, there follows in Chapter 5 a geometrization of certain second order differential equations. Limiting himself first to two dimensions the author shows that with each equation of the form

$$\frac{d^2v}{du^2} = A\left(\frac{dv}{du}\right)^3 + 3B\left(\frac{dv}{du}\right)^2 + 3C\frac{dv}{du} + D,$$

with $A, \cdots, D$ functions of u, v, there can be invariantively associated a certain "normal" projectively connected space having this equation for the equation of its geodesics. The generalization of this result to n dimensions leads to a connection identical with the one associated with a system of paths by T. Y. Thomas. As an illustration of the application of projective methods to metric geometry there is a discussion of geodesic representation of surfaces in euclidean three-space using the projective connection associated with the geodesics.

Chapter 6 is devoted to the study of a few typical problems concerning surfaces in projectively connected three-space. For example, it is proved that the vanishing of a certain tensor, explicitly given, is a necessary and sufficient condition that every surface of the space have reciprocally conjugate tangents. Considerable attention is given to the problem of generalizing the tangents of Darboux, with the conclusion that several alternative definitions are possible leading to distinct configurations.

The final chapter approaches projectively connected spaces through their holonomic group, a notion which has been one of the author's most fruitful contributions to differential geometry. This approach, unfamiliar to many, is treated with unusual clarity. General directions are given for attacking the two fundamental problems: (1) given a projectively connected space, to find its holonomic group, and (2) given a group, to determine the possible spaces having it for holonomic group. Problem 2 is solved for several important groups. Last comes a determination of all holonomic groups of "normal" projective two-spaces. There is a bibliography which seems short and rather haphazard.

Of all Cartan's books this is one of the most clearly written but, like the others, it can be fully appreciated only by those having some previous acquaintance with its field. The author works through his subject informally, enriching it with his unified point of view and his unsurpassable geometric insight, finding new approaches leading to a better understanding, and giving here and there a new result for good measure. This is probably the first major publication in book form to have classical and generalized projective differential geometry fraternizing within its covers. And the author was a pioneer in both fields.

J. L. VANDERSLICE

REMES ON APPROXIMATION

On Methods for Obtaining the Best Approximation of Functions, in the Sense of Tchebycheff. By E. Remes (in the Ukrainian language). Mathematical Institute of the Ukrainian Academy of Sciences, Kieff, 1935. i+159 pp.

This is one of a series of mathematical monographs being published in Soviet Russia. Its main object is the study of polynomial—more particularly, "best" polynomial, in the sense of Tchebycheff—approximation of various classes of functions. Considerable space is devoted to functions of several variables—a fertile field awaiting further cultivation. Following the general utilitarian tendencies of science in Soviet Russia, the theoretical discussion is in places rather condensed. To compensate, methods are developed for the actual computation of the approximating polynomials, which are applied to concrete examples, the error involved being also given. Thus the methods and results are made usable in engineering practice.

The book is divided into two parts. Part I deals with problems in the theory of approximation which admit solutions in a closed form. It consists of the following five sections: §1. Approximate representation of a function $f(x, y, \cdots, v)$ defined in a certain region, by means of linear functions $\phi_1=\alpha_0+\alpha_1x+\cdots$ or functions of the type $\phi_2=\alpha_0+\alpha_1x+\cdots+\alpha_{1,2}xy+\cdots+\alpha_{1,2}\ldots xy\cdots v$ (choice of the above parameters, relative and absolute error). Application to harmonic, sub- and super-harmonic functions ($\Delta^2f\equiv\partial^2f/\partial x^2+\cdots+\partial^2f/\partial v^2=0$, $\geqq 0$, $\leqq 0$ in a certain region). Special consideration is given to Tchebycheff's Map-Problem which essentially reduces to the following one: Find a harmonic function $\Phi(x, y)$ with the least possible deviation from $f(x, y)=2/(e^x+e^{-x})$ in a certain region bounded by a continuous contour C. Show that $\Phi(x, y)$ differs by a constant from that harmonic function which on C takes on values given by the above $f(x, y)$. §§2, 3. Approximate representation of functions w defined by equations of the type $w^\rho=x^\rho+y^\rho+\cdots+v^\rho$, that is,

$$(x^\rho + y^\rho + \cdots + v^\rho)^{1/\rho} \leqq \alpha x + \beta y + \cdots + \kappa v.$$

The case $\rho=2$ deserves—and receives—special consideration. In fact, this is the celebrated Poncelet problem: Find the best possible linear approximation for $(x^2\pm y^2)^{1/2}$, $(x^2+y^2+z^2)^{1/2}$; more generally, for $(x_1^2+\cdots+x_n^2)^{1/2}$. It was this problem which in the masterly hands of Tchebycheff became the cornerstone of the theory of best approximation. The book deals with this problem in §§2 and 3, analytically and geometrically, for $n=1, 2, \cdots, 10$. A few results follow, as an illustration.

$$(x_1^2 + x_2^2)^{1/2} \leqq .960x_1 + .398x_2, \qquad \text{within } 4\%,$$

$$(x_1^2 + x_2^2 + x_3^2)^{1/2} \leqq .940x_1 + .328x_2 + .299x_3, \qquad \text{within } 6\%,$$

$$x_1 \geqq x_2 \geqq \cdots \geqq x_{10} \geqq 0.$$

$$(1 + y^2)^{1/2} \leqq .955 + .414y, \quad \text{absolute error} < .045, \qquad 1 \geqq y \geqq z \geqq 0.$$

$$(1 + y^2 + z^2)^{1/2} \leqq .926 + .414y + .318z, \quad \text{absolute error} < .074.$$

§4. Linear approximations for algebraic rational fractions and irrational functions. Application is then made to differential equations (illustration: $dy/dx=x^{1/2}+y^{1/2}$). The relation of the above considerations is shown—rather sketchily—to Bernoulli's method for obtaining that root of an algebraic equation whose modulus is largest. The formulas given in §2 are further applied to Graeffe's method for solving algebraic equations. This is developed with more details and computations. §5. The best ap-

proximation to a convex function by means of the ordinates of a polygonal line, with a preassigned number of vertices. Best linear approximation for x^p, xy, xyz, $\cdots$. Approximation of transcendental functions and solution of transcendental equations. Approximation of functions by means of the above functions ϕ_2. Thus,

$$e^x \leqq .8459x^2 + .9549x + 1.0007, \quad \text{error} < .088, 0 \leqq x \leqq 1,$$
$$e^{(x+y)/2} \leqq .946 + .649(x + y) + .421xy, \quad \text{error} < .054, 0 \leqq x, y \leqq 1,$$

and, with the last term "linearized,"

$$e^{(x+y)/2} \leqq .908 + .638x + 1.050y. \tag{1}$$

Part II deals with general methods for obtaining the "best" polynomial approximation for functions in one or several variables. It consists of three sections: §6. Characterization and general properties of the polynomial of best approximation to a function $f(x)$ defined on a given set $E<(a, b)$; existence, bounds for the best approximation, $\cdots$. The discussion follows in the main that of the well known book of de la Vallée-Poussin on approximation. §7. Application of the method of successive approximations in order to obtain the polynomial in question. The author develops two algorithms. The second seems to us to be the more interesting one. It is based upon first finding the polynomial of best approximation, of degree $\leqq n$, to $f(x)$—assumed to be given on (a, b)—corresponding to $n+2$ points arbitrarily chosen on (a, b), and then introducing proper corrections. The above methods are applied to $f(x) = |x|$, $|x| + \kappa \operatorname{sgn} x$, $\kappa > 0$. Thus, the polynomial $.0675 + 1.9313x^2 - 1.0665x^4$ represents $|x|$ on $(-1, 1)$, within 0.0677; the difference between this approximation and the best possible one (the degree of the polynomial being $\leqq 4$) is $<10^{-4}$. §8. This section deals with systems of linear equations and with the best linear approximation of functions in several variables. The author emphasizes the importance of this section as was stated above. The method of successive approximations is sketched. Use is made of the results obtained in Part I. Thus, in order to obtain the best linear approximation to $e^{(x+y)/2}$ in the region $1 \geqq x \geqq y \geqq 0$ we start with the above approximation (1), and seek to determine the necessary corrections. In this way we find that the best approximation ρ has the value $\rho = .106$, (error of order 10^{-4}).

The book closes with a treatment of the best approximation of empirical functions (given by a set of data) by means of linear aggregates of functions of preassigned type.

J. A. SHOHAT

SHORTER NOTICES

Leçons sur les Séries Hypergéométriques et sur quelques Fonctions qui s'y Rattachent. I. *Propriétés Générales de l'Équation d'Euler et de Gauss.* By É. Goursat. (Actualités Scientifiques et Industrielles, no. 333.) Paris, Hermann, 1936. 92 pp.

Within a comparatively limited space and assuming only the fundamental conceptions about linear differential equations in the complex domain, the author develops the principal facts about the hypergeometric equation

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0.$$

We find the representation of Kummer's twenty-four integrals in terms of hypergeometric series, provided the numbers γ, $\gamma-\alpha-\beta$, $\alpha-\beta$, are not integers; the representation of these integrals in form of curve integrals (following a general method of Euler); a complete discussion of the case in which the numbers mentioned before are integers; the study of the group of the hypergeometric equation, and also the determination of certain linear relations between various integrals.

The discussion of the logarithmic case is particularly valuable; it can not be found easily in the usual textbooks on differential equations.

In the formula (13) on p. 54 the term $(z^2-1)^n$ has to be replaced by $(1-z^2)^n$.

The present pamphlet is the first part of a monograph planned on hypergeometric series. Unfortunately the author died on November 26, 1936.

G. Szegő

Functions of Real Variables. By William Fogg Osgood. University Press, National University of Peking, 1936. 12+399 pp.

Functions of a Complex Variable. By William Fogg Osgood. University Press, National University of Peking, 1936. 8+257 pp.

These two texts are prepared for the student who has completed a course in "advanced calculus," as for example, one based upon the well known text, by the author, and who is about to enter into the deeper mysteries of mathematical analysis. The second of the two works under review presupposes also some knowledge of functions of real variables but does not require as much as is handled in the first of these two volumes. One finds here well-organized courses, systematic, lucid, fundamental, with many brief sets of appropriate exercises, and occasional suggestions for more extensive reading. The technical terms have been kept to a minimum, and have been clearly explained. The aim has been to develop the student's power and to furnish him with a substantial body of classical theorems whose proofs illustrate the methods and whose results provide equipment for further progress. There is throughout a wholesome regard for steady application to essentials, with no vague references to diverting side issues. There is no room here for discussion of such special topics of increasing modern interest for students of real variables as summability of general divergent series, systems of orthogonal functions, or abstract spaces. Even the theory of Lebesgue measure is left for later study. So also in the briefer course in complex variables, Hadamard's three circles theorem and the discussion of the maximum modulus are of course not mentioned. Even the classical topic of elliptic functions claims less than ten pages. The lack of index to either volume seems an unnecessary handicap.

For the prospective teacher or student looking for a sound investment for his

mathematical efforts the present works may be recommended. Perhaps the most pertinent information that this review could contain would be a brief outline of the contents of these respective volumes.

Functions of Real Variables. Chapter 1 deals with convergence of infinite series, including Kummer's criterion, a discussion of infinite products, and a very brief account of the hypergeometric series. Having thus whetted the reader's appetite for the study of limits, the author devotes Chapter 2 to the number system. Point sets, limits, and continuity constitute the subject matter of Chapter 3, which opens with fifteen examples of point sets, and proceeds to the covering theorem (usually called the Heine-Borel theorem), and the axiom of choice. The next two chapters give the classical theory of derivatives, integrals, implicit functions, and uniform convergence. Chapter 6 deals with the "elementary functions," trigonometric and exponential, discussing very briefly their important properties as functions of real variables, including mention of infinite product representations. In the next chapter, infinite series are studied systematically with regard to their algebraic manipulation. The previous chapters should serve to make the theory concretely applicable and further applications of interest are introduced, in particular the Legendre polynomials and the Bernoulli numbers. Chapter 8 gives a brief but substantial study of the basic ideas connected with Fourier series, even including Gibbs' phenomenon. The discussion of point sets, continuity, uniform convergence, Jordan curves, and other such topics given previously, is applied in Chapter 9 to the theory of definite integrals, including line integrals. Applications are made to classical integrands, "Duhamel's Theorem" is critically discussed, and the student is by this time fairly started in classical analysis. The final chapters are devoted to systematic study of the gamma function, the Fourier integral, and the existence theorems for solutions of differential equations.

Functions of a Complex Variable. The contents suggest in large part a simplification of the more elaborate treatise of the author on this same subject, printed in the German language. Chapters 1, 2, and 3 deal with the algebra and geometry of complex numbers and linear (fractional) transformations, with a study of the definition of analytic functions giving applications to some elementary functions. Riemann surfaces, illustrated by some very simple cases, constitute the subject of Chapter 4. The Cauchy theory leading to the Cauchy-Taylor development and disposing, in passing, of the "fundamental theorem of algebra," provides the material for Chapter 5. The further development of the theory of analytic functions as due chiefly to Weierstrass and Riemann, together with two more proofs of the "fundamental theorem of algebra," is given in Chapter 6. The chapter includes a treatment of residues, and further material on conformal mapping. Darboux's theorem is given and applied to the theory of the mapping of a rectangle on a circle, thus giving insight into an aspect of the elliptic integrals, although their name is nowhere mentioned in this chapter. Chapter 7 on "analytic continuation" discusses also the problem of uniformizing functions, in particular algebraic functions, and gives a survey of elliptic functions as approached through elliptic integrals. The last two chapters deal with the logarithmic potential and the conformal mapping of a simply connected region, with Picard's theorem a high point of the discussion.

The books are neatly printed in English and manifest the results of extraordinarily painstaking press work on the part of the Chinese publishers, and careful proofreading by the author. Save for the inevitable handicap of studying a subject in a foreign language, Chinese students may well be grateful for texts so clear, cogent, and fundamental.

A. A. BENNETT

Les Fonctions Polyharmoniques, by Miron Nicolesco. (Actualités Scientifiques et Industrielles, no. 331; exposés sur la théorie des fonctions, IV.) Paris, Hermann, 1936.

This monograph is the fourth in a series devoted to recent developments in the theory of functions, which is being published under the editorship of Professor Montel of Paris. These monographs have as an objective the presentation of synthetic summaries of some of the advances made during the last few years; no attempt is made at giving a detailed exposition, but a clear-cut sketch of the developments, supplemented by suitable bibliographies, is made available to the reader who wishes to orient himself in the fields treated.

The polyharmonic functions of order p (or p-harmonic), in an n-dimensional space, are those functions $U(x_1, x_2, \cdots, x_n)$ of n independent variables which satisfy the partial differential equation $\Delta^p U = 0$, where Δ^p is the pth iteration of the Laplacian operator Δ. Thus,

$$\Delta \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \Delta^k = \Delta(\Delta^{k-1}), \quad (k = 1, 2, 3, \cdots), \quad \Delta^0 \equiv 1.$$

For $p=1$, we have the case of the ordinary harmonic functions. The biharmonic case ($p=2$), being of importance in the theory of elasticity, was the object of research rather early. Thus, Airy introduced the stress function from which the components of the stress tensor may easily be calculated, and showed that it had to satisfy the equation $\Delta^2\phi = 0$, with appropriate boundary conditions.

The earlier researches on polyharmonic functions centered, quite naturally, about the study of boundary value problems analogous to the classical ones in the theory of harmonic functions. Most of the results obtained in this direction are due to the Italian school of mathematicians led by Almansi, Volterra, Marcolongo, Lauricella, Boggio, and others. In these researches, the investigation of the intrinsic, structural properties of the polyharmonic functions was, for the most part, subsidiary to the solution of the particular boundary value problems being studied. Even so important a result as Almansi's expansion theorem was treated primarily from the point of view of its bearing on this type of problem.

Since 1930, Professor Nicolesco and others have undertaken a systematic study of the intrinsic properties of the polyharmonic functions for their own sake and have succeeded in filling important gaps in the theory of these interesting functions. The main purpose of his monograph is to give a synthesis of the results so far obtained. The work also contains a brief summary of some of the work done in connection with these functions and their associated boundary value problems, placing some emphasis on the biharmonic case of elasticity theory.

An extensive memoir by Professor Nicolesco, entitled *Recherches sur les fonctions polyharmoniques*, to appear shortly in the Annales de l'École Normale Supérieure, as a supplement to the present work, should be of the greatest interest.

M. A. Basoco

Les Conditions de Monogénéité. By D. Menchoff. (Actualités Scientifiques et Industrielles, no. 329.) Paris, Hermann, 1936. 53 pp.

This monograph, as its title implies, is a report on some of the recent developments in the direction of obtaining sufficient conditions that a function of a complex variable be monogenic at a point or holomorphic in a region. Monogeneity at a point being equivalent to the existence of a finite-valued derivative at the point, the initial

theorem quoted gives also the simplest condition, which is that the functions u and v, where $f=u+iv$, possess a Stolz differential and satisfy the Cauchy-Riemann differential equations. If the Stolz differential is assumed, then it is also sufficient either (a) that the difference quotient $\Delta f/\Delta z$ have the same limit for any two distinct directions, or (b) that arg $\Delta f/\Delta z$ have the same limit for three distinct directions, or (c) that $|\Delta f/\Delta z|$ have the same limit in three directions, but in the latter case f may be monogenic instead of f. The theorems giving sufficient conditions for the holomorphism of a function in a region depend upon sufficient conditions for the expression of the integral $\int f(z)dz$ around a rectangle with sides parallel to the axes in the form of the double integral

$$-\iint\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)dx\,dy+i\iint\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dx\,dy$$

and the fact that if $\overline{\lim}\,|\Delta f/\Delta z|$ is finite on a measurable set, then the Stolz differential of f exists except on a set of measure zero, various theorems being obtained by giving sufficient conditions for the application of these results and the use of Morera's theorem.

In the main the monograph is a brief presentation of the author's investigations in these questions, which may be justifiable, but involves the possibility of overlooking simplifications in presentation. For instance, the three supplementary sufficient conditions for the monogeneity of a function at a point are geometrically intuitive if use is made of the Kasner circle.* However, the monograph is informative and suggestive; especially might one call attention to the remark in the introduction that, while many theorems have a form which involves only the complex variable situation, it has so far been necessary to use in their proof deep-seated methods of the modern theory of real functions, and that it would be interesting and desirable to derive these same theorems without departing from the setting in which they are stated.

T. H. HILDEBRANDT

Théorie de l'Addition des Variables Aléatoires. By P. Lévy. (Monographies des Probabilités, publiés sous la direction de E. Borel, no. 1.) Paris, Gauthier-Villars, 1937. 17+328 pp.

Professor Lévy's book is the first of a series of monographs on probability and its applications. This first monograph covers the field of Lévy's special interest, to which he has made valuable and often definitive contributions: certain questions connected with asymptotic problems in probability theory. The first four chapters include a general introduction to probability, designed to make the book complete in itself. Chapter V contains theorems related to the Gaussian law—Cramer's recently proved theorem and various extensions of the Liapounoff theorem. In Chapter VI, Lévy discusses series whose terms are mutually independent chance variables. He includes an important theorem on dispersion, recently derived by himself and Doeblin. Chapter VII considers a chance variable x_t (depending on the parameter t) whose increments in non-overlapping t-intervals are independent. The treatment is somewhat confusing in that measurability considerations are omitted and it is not clear whether x_t considered as a function of t has been proved to have (almost certainly), at worst, jumps at its points of discontinuity. Lévy gives a considerably simplified version of

* This Bulletin, vol. 34 (1928), p. 561.

his derivation of the general form of the characteristic function of an indefinitely divisible distribution law. He then goes on to consider stable distribution laws and related topics. Chapter VIII extends various results (Liapounoff theorem, law of iterated logarithm, and so on) known for sets of mutually independent chance variables, replacing the condition of independence by certain more general conditions. In Chapter, IX Lévy discusses measure properties of developments in continued fractions, using the suggestive terminology of probability.

Lévy's book can be recommended only to advanced students of probability who already have some familiarity with the topics treated. Other readers will merely be exasperated by his confidence that they have his own unsurpassed intuitive grasp of the subject. The student preparing himself to do research in probability, however, will find here the latest results in an important field, derived in a way which stresses methods rather than details.

J. L. Doob

Les Lois des Grands Nombres du Calcul des Probabilités. By L. Bachelier. Paris, Gauther-Villars, 1937. 7+36 pp.

Bachelier believes that many of the results in his books and papers have been unnoticed by later writers. In this book he restates many of these results (without proofs). He first considers the Bernoulli case: independent trials, each having only two possible results, having probability p and $1-p$. He then generalizes in various directions, letting p vary from trial to trial, and so on. The formulas are of asymptotic character, approximations which improve as the number of trials increases. As an example of their general character, we give one result. Bachelier finds that (in the Bernoulli case) if μ_1 trials are made, and if we consider the difference between the number of times the event with probability p has occurred and its expected value, then the probability that this difference will return to 0 before μ further trials are made is $(2/\pi)$ arc tan $(\mu/\mu_1)^{1/2}$.

J. L. Doob

Theorie der Orthogonalreihen. By Stefan Kaczmarz and Hugo Steinhaus. (Monografje Matematyczne, vol. VI.) Warsaw, 1935. vi+298 pp.

The present volume of the excellent Polish Series is devoted to the theory of general orthogonal functions of a single real variable. Desiring not to increase the size of the volume without proportionally increasing its usefulness, the authors omitted almost completely the theory and applications of special orthogonal functions including that of orthogonal polynomials, and concentrated their attention on general orthogonal functions as a tool in pure mathematics. Even in this restricted field no claim is made for "encyclopaedic completeness." Despite these somewhat severe restrictions the authors succeeded in presenting a very interesting material widely scattered in the literature, including also some new contributions of their own.

The book consists of eight chapters followed by a bibliography containing 129 items. Chapter 1 (pp. 1–30) gives a brief exposition of general notions of abstract spaces, and linear operations and functionals which serve as a most important tool in the subsequent developments. Chapter 2 (pp. 37–60) introduces the fundamental concepts of orthogonality, completeness, closure, and best approximation. Chapter 3 (pp. 61–102) discusses general orthogonal series in L^2 including theorems of Müntz and of Riesz-Fischer, and Parseval's identity. Chapter 4 (pp. 103–148) treats of various examples, with particular attention given to orthogonal systems of Haar and

of Rademacher and to the "probabilistic" interpretation of the latter. Chapter 5 (pp. 149–194) is devoted to the theory of convergence (almost everywhere, unconditional, · · ·), divergence, and summability of orthogonal series. The climax of this chapter is reached in an elegant proof of the fundamental theorem of Rademacher-Menchoff. A systematic use of "Lebesgue's functions" associated with orthogonal expansions deserves a special mention. Chapter 6 (pp. 195–242) deals with orthogonal expansions in various spaces (L^P, C, M) different from L^2. Among various topics treated here we mention the relationships between the closure and completeness of an orthogonal system; the theorems of Young-Hausdorff and of Paley; the theory of "multipliers" transforming orthogonal expansions of functions of various classes into each other; and a discussion of various singularities which occur in orthogonal expansions. Chapter 7 (pp. 243–260) reveals various remarkable properties of "lacunary" series. Chapter 8 (pp. 261–298), the last chapter, is of somewhat mixed character, being devoted partly to biorthogonal expansions, and partly to polynomials orthogonal relative to a given weight-function.

The exposition, which is in general clear and concise, in some places shows a tendency to be either somewhat vague, or so condensed that it will be difficult to follow for a reader who is not well versed in the field. The number of misprints (in addition to those mentioned in a list of 16 Errata) and of slips of pen or thought is not entirely negligible. Thus on page 6 the reader is told that every point set can be decomposed into a sum of a perfect set and of an at most denumerable set; on page 19 the norm of the functional $\int_a^b dg$ is stated to be $\int_a^b |dg|$; the condition [874] on page 280, which is sufficient for the completeness of the system of orthogonal polynomials relative to the weight function $w(t)$, is obviously not satisfied in the case of Laguerre polynomials, contrary to the assertion preceding this condition. In order to avoid footnotes the authors are using a new scheme of cross references, which, according to the reviewer's experience, does not represent an improvement over the customary system.

J. D. TAMARKIN

Introduction à la Théorie des Fonctions de Variables Réelles. Parts I and II. By Arnaud Denjoy. (Actualités Scientifiques et Industrielles, nos. 451 (55 pp.) and 452 (57 pp.).) Paris, Hermann, 1937.

These are two of the brochures in the section on *Sets and Functions*, under the editorship of Denjoy, who here writes Parts I and II on the introduction to real function theory. Having himself gained important results (on derived numbers, totalization, and so on), Denjoy has paused to scan the field of sets and real variables. From the brevity of each brochure it is clear that his treatment of the various topics is necessarily skeletal (it omits all proofs), but, we believe, is interesting and successful.

Early in Part I he advances reasons of a physical nature for studying non-analytic, even discontinuous, functions. Then comes a foremention of names and topics to be considered: Cauchy, Abel, Riemann (on convergence and integration), Cantor (sets, transfinite numbers), Baire (classification of functions), Borel, Lebesgue (measure and integration), · · · ; and a mention of general analysis. Chapter 2 deals with the geometry of Cartesian point sets, that is, familiar point set theory (including measure). The author here makes distinction between descriptive (topological) ideas and metric ideas, which distinction is also carried over to Chapter 3 on the analysis of functions. Examples of descriptive notions are continuity, convergence, the Baire classification; of metric notions, derived numbers, differentials. Functions of

bounded variation illustrate a combination of the two ideas. Brief mention is also made of the theorem on polynomial approximation to continuous functions, of Tchebycheff polynomials, and of summability of sequences.

Part II, beginning with integration, applies the theory. Brief treatments of the Riemann, Lebesgue, and Stieltjes integrals are given, and mention is made of the Denjoy totalization process. Chapter 2 takes up trigonometric series. The importance of such series in leading to a general conception of function (attributed to Riemann) is indicated. There follow remarks concerning the determination of the coefficients, uniqueness, tests for convergence, the Fejér method of summation; then a short note on general orthogonal functions. Chapter 3 deals with quasi-analytic functions (dear to the heart of Denjoy). He sets the general problem leading to such functions, and states the solution of Carleman. A last chapter discusses functionals: continuity, linearity, differential of a functional, functions of sets.

Denjoy had looked upon the handiwork of those who have made sound contributions to real variable theory, and saw that it was good.

I. M. SHEFFER

Sur les Fonctions d'une Variable Complexe Représentables par des Séries de Polynômes. By M. Lavrentieff. (Actualités Scientifiques et Industrielles, no. 441.) Paris, Hermann, 1936. 60 pp.

This brochure is in the section of its series on *The Theory of Functions*, edited by Paul Montel. Baire found the condition that a function of a real variable be of class one (representable as a series of continuous functions). Using the approximation of continuous functions by polynomials (Weierstrass), one may say that Baire determined when a function is the sum of a series of polynomials. The extension of this problem to the complex plane is the subject of Lavrentieff's work. He treats the problem from its classical beginnings to the latest advances, thus providing a valuable résumé.

Chapter 1 is introductory. Baire's work is taken as starting point, after which the first general advance to the complex plane, the well known Hilbert-Runge theorem, is stated and proved. In the short second chapter are stated some needed theorems on the correspondence of boundaries under conformal mapping.

One of the problems dealt with in Chapter 3 concerns the convergence of sequences of (holomorphic, in particular, polynomial) functions that are bounded in their set, together with a uniqueness theorem (to the effect, for example, that two functions which are limits of such sequences are identical in a given region if they coincide on certain point sets of the boundary). Another problem treated is that of finding necessary conditions and sufficient conditions that a function have a polynomial expansion (of given type) on a closed point set. An extension to harmonic sequences is indicated.

In the last chapter special point sets M and M^* (which space does not allow to describe adequately) are introduced, and it is shown that these sets are important in the statement of general theorems on the representation of holomorphic functions by series of polynomials.

I. M. SHEFFER

Diophantische Approximationen. By J. F. Koksma. (Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 4.) Berlin, Springer, 1936. 8+157 pp.

It is remarkable material on diophantine approximations with which the author presents us in this volume. The bibliography, which seems to be unusually complete,

contains about eight hundred separate titles. It is needless to emphasize the importance of such a survey for those working in this field.

Let us only mention a few of the main theories considered by the author. The theory of diophantine approximations has been taken in its widest sense, so that one also obtains information about neighboring theorems and applications. The book begins with a short survey of the field to obtain a classification of the main problems. One then finds a discussion of Minkowski's analysis, his Geometry of Numbers and applications to homogeneous and non-homogeneous linear forms. At this point one should also mention the account of Blichfeldt's methods. The properties of continued fractions are then discussed together with the neighboring theories, namely, of irrationality and transcendency and their measures. The approximation theorem of Kronecker is discussed at length. The distribution of real numbers (mod 1) covers two chapters and one finds accounts of the investigations of Weyl, Vinogradoff, van der Corput, and others. The final chapter is devoted to the theory of diophantine inequalities.

OYSTEIN ORE

Neuere Fortschritte in den exacten Wissenschaften. Fünf Wiener Vorträge. Ser. 3. *Vitamine und ihre Bedeutung.* By E. Späth. *Die physikalischen Entdeckungen der letzten Jahre.* By H. Thirring. *Extreme Versuchsbedingungen als Quelle des Fortschrittes.* By H. Mark. *Prinzipielle Fragen in der modernen Physik.* By W. Heisenberg. *Einige neuere Fortschritte in der exacten Behandlung sozialwissenschaftlicher Probleme.* By K. Menger. Leipzig and Vienna, Franz Deuticke, 1936. 132 pp.

Five scholarly lectures by experts in physiology, physics, and social science are here presented. The first, on vitamins, is good but not novel. Professor Thirring's exposition of the discovery of the elementary particles of physics is systematic and up to his usual high standard. The third lecture is especially timely for mathematicians and theoretical scientists since it gives an account of the precision of the experimental determination of the major empirical constants which enter into physical theories. Perhaps the most unique and interesting of the lectures is by Heisenberg. He argues against a future reduction of the statistical aspects of quantum theory to causal laws. The final essay offers suggestive and original evidences of the application of mathematics and mathematical logic to social science. These lectures maintain a high level of excellence and merit the attention of serious students.

F. S. C. NORTHROP

NOTES

In addition to the Editorial Board of the Transactions of this Society, the following persons have assisted the editors by refering papers offered for publication in the volumes for 1937: H. R. Brahana, E. W. Chittenden, A. B. Coble, H. B. Curry, E. L. Dodd, J. L. Doob, B. O. Koopman, Saunders MacLane, John von Neumann, J. B. Rosser, E. B. Stouffer, Otto Szász, S. M. Ulam, R. L. Wilder. The editors appreciate this assistance and desire to give it public recognition.

A Mathematical Colloquium, similar to those held in 1926, 1930, and 1934, will be held in St. Andrews, Scotland, from July 4 to July 15, 1938, under the auspices of the Edinburgh Mathematical Society. Short courses of lectures will be given by prominent mathematicians on topics of pure and applied mathematics. On their way to the Colloquium, members will have the opportunity of attending a meeting of the Royal Society of Edinburgh, celebrating the three hundredth anniversary of the birth of James Gregory. This will be held in Edinburgh on July 4. For particulars concerning membership of the Colloquium, apply to the Hon. Secretary, Edinburgh Mathematical Society, 16 Chambers Street, Edinburgh 1.

The University of Notre Dame announces a symposium on the algebra of geometry and related subjects to be held on February 11 and 12. Among those who will participate are A. A. Albert, Emil Artin, Garrett Birkhoff, E. V. Huntington, Georges Lemaître, Karl Menger, John von Neumann, Oystein Ore, and M. H. Stone.

At the recent celebration of the hundredth anniversary of the founding of the University of Athens, honorary doctorates were conferred on Dean G. D. Birkhoff, of Harvard University, and Professor E. L. Thorndike, of Columbia University.

At the annual dinner of the Radio Club of America held in New York on October 29, 1937, the Armstrong medal and citation for 1937 was presented to Dr. Alan Hazeltine, professor of mathematics at Stevens Institute of Technology, "in recognition of his outstanding contributions to the art and science of radio communication." He is the first to receive the award, which was established by the Radio Club in 1935 in honor of the achievements of Major E. H. Armstrong, professor of electrical engineering at Columbia University.

Dr. Wolfgang Haack, of the Technical School of Berlin, has been appointed to a professorship at the Technical School of Karlsruhe.

Professor J. B. S. Haldane has been appointed first incumbent of the chair of biometry at the University College, London.

Assistant Professor Rudolf Iglisch, of the Technical School of Aachen, has been appointed to a professorship at the Technical School of Braunschweig.

Assistant Professor Fritz Lettenmeyer, of the University of Munich, has been appointed to a professorship at the University of Kiel.

Assistant Professor Franz Rellich, of the University of Marburg, has been promoted to a professorship in applied mathematics.

Assistant Professor Robert Sauer, of the Technical School of Aachen, has been promoted to a professorship in applied mathematics.

Dr. Herbert Seifert, of the Technical School of Dresden, has been appointed to a professorship at the University of Heidelberg.

Professor R. B. Allen, of Kenyon College, is on leave of absence during 1937–38. He is spending the year in Paris.

Dr. Reinhold Baer has been appointed to an assistant professorship at the University of North Carolina.

Dr. A. H. Diamond, of the University of California at Berkeley, has been appointed lecturer in mathematics for 1937–38 at the University of California at Los Angeles.

Assistant Professor W. L. Duren, of Tulane University, has been promoted to an associate professorship.

Professor W. B. Fite, of Columbia University, will be absent on leave during the winter session. During his absence Professor J. F. Ritt is acting as executive officer of the department of mathematics.

Mr. John Gaffney, of the Michigan College of Mining and Technology, has been promoted to an assistant professorship.

Assistant Professor L. L. Garner, of the University of North Carolina, is on leave of absence for the session of 1937–38.

Associate Professor G. I. Gavett, of the University of Washington, has been promoted to a professorship.

Dr. F. C. Gentry, of the University of Illinois, has been appointed to an assistant professorship at the University of Oklahoma.

Dr. S. G. Hacker has been appointed to an assistant professorship at Washington State College.

Associate Professor Cora B. Hennel, of Indiana University, has been promoted to a professorship.

Assistant Professor A. S. Householder, of Washburn College, has leave of absence for the year 1937–38. He is spending the year at the University of Chicago.

Assistant Professor V. A. Hoyle, of the University of North Carolina, has been promoted to an associate professorship.

At the University of Illinois, Dr. D. G. Bourgin, Dr. J. L. Doob, and Dr. P. W. Ketchum have been promoted to assistant professorships, while assistant professors J. H. Miles and G. E. Moore have been promoted to associate professorships.

Mr. I. F. Keeler, of the University of Iowa, has been appointed to a professorship at North Central College, Naperville, Ill.

Professor C. G. Latimer, of the University of Kentucky, will be a visiting lecturer at the University of Wisconsin during the second semester.

Mr. Sam Legvold has been appointed to an assistant professorship in physics at Luther College, Decorah, Iowa.

Mr. W. A. Longacre, of the Michigan College of Mining and Technology, has been promoted to an assistant professorship.

Dr. G. C. Munro has been appointed to an assistant professorship at the Kansas State College.

Dr. J. F. Randolph, who is spending his second year at the Institute for Advanced Study, has been appointed associate professor of mathematics at the University of Oklahoma, his duties to begin in September, 1938.

Assistant Professor W. E. Roth, of the University of Wisconsin at Milwaukee, has been made an associate professor.

Dr. O. K. Sagen, of Iowa State College, has been appointed to an assistant professorship at the University of Maryland.

Dr. W. E. Sewell, of the Georgia School of Technology, has been promoted to an assistant professorship.

Dr. S. R. Smith, of the Carnegie Institute of Technology, has been made an assistant professor.

Associate Professor H. E. Stelson, of Kent State University, has been promoted to a professorship.

Professor W. M. Whyburn, acting chairman of the department of mathematics at the University of California at Los Angeles, has been made chairman of that department.

Associate Professor R. B. Wildermuth, of Capital University, has been promoted to a professorship.

The following appointments to instructorships are announced: Agricultural and Mechanical College of Texas: Mr. R. R. Lyle; Armour Institute of Technology: Dr. D. G. Fulton, Dr. I. E. Perlin; Brown University: Dr. Douglas Derry; University of California at Los Angeles: Dr. W. C. Risselman; University of Cincinnati: Dr. Paul Pepper; Cornell University: Mr. Edwin Galbraith, Mr. W. J. Harrington, Mr. Karl Johannes, Mr. D. S. Miller, Mr. Seymour Sherman, Mr. G. B. Thomas, Mr. W. D. Wray; Hunter College: Dr. L. A. Knowler; University of Illinois: Dr. Josephine H. Chanler, Dr. Ruth G. Mason; University of Iowa: Dr. E. N. Oberg; Johns Hopkins University: Dr. D. L. Netzorg; Kansas State College (part-time): Mr. Harold Wierenga; Lafayette College: Mr. F. A. Ficken; Lehigh University: Mr. H. W. Alexander; Massachusetts Institute of Technology: Mr. W. R. Hydeman, Mr. S. W. Stewart; Michigan College of Mining and Technology: Mr. V. O. York; University of Nebraska: Mr. J. D. Novak; University of Nevada: Mr. Ingo Maddaus; New Mexico State College of Agriculture and Mechanic Arts: Dr. Gordon Fuller; Oregon State College: Dr. P. G. Hoel; Purdue University: Dr. A. H. Bailey; University of Rochester: Dr. A. S. Galbraith, Dr. S. E. Warschawski; Rose Polytechnic Institute: Mr. T. P. Palmer; Southern Methodist University: Dr. K. L. Palmquist; Virginia Military Institute: Mr. I. G. Foster; Washburn College: Mr. Paul Eberhart.

Dr. E. W. Paxson, who was reported in the November issue of this Bulletin, page 763, to hold an instructorship in the California Institute of Technology, has informed

the editors that this announcement is incorrect. As stated in the September issue, page 611, Dr. Paxson holds an instructorship in Wayne University.

The death of Dr. Otto Hölder, formerly professor of mathematics at the University of Leipzig, is reported. He was seventy-eight years old.

Lord Rutherford, director of the Cavendish Laboratories at Cambridge University and famous for his researches in radioactivity, died on October 19, 1937, at the age of sixty-six years.

Dr. L. M. Hoskins, professor emeritus of applied mathematics at Stanford University, died on September 8, 1937, at the age of seventy-seven years. He had been a member of the Society for thirty-five years.

Professor Frank Morley, former President of this Society, since 1928 emeritus professor of mathematics at the Johns Hopkins University, died on October 17, 1937, at the age of seventy-seven years. A more adequate notice will appear in a later number of this Bulletin.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. A. T. Craig: *On correlation due to common elements.*

In this paper linear functions of independently observed values of a chance variable x are considered when these functions have in common certain of the observed values. A necessary and sufficient condition that the functions shall have a linear regression system is derived. An investigation is also made of the regression system of real symmetric quadratic forms of normally and independently distributed variables. (Received November 19, 1937.)

2. H. L. Rietz: *On the correlation of a mean and standard deviation in small samples drawn from a certain non-normal population.*

The parent distributions with which this paper is concerned are given by urn schemata devised by the author some years ago to give meaning to measures of correlation in relation to certain given probabilities. The theoretical distributions resulting from the urn schemata were published in the Annals of Mathematics, vol. 21 (1920), pp. 306–322. In 1925, Leone E. Chesire prepared a master's thesis relating to small samples drawn from these non-normal distributions. Certain data that were obtained by the experimental sampling of Miss Chesire seem to be appropriate material for the study of the correlation of mean and standard deviation, and the corresponding regressions. The main object of the present paper is to report certain results of such a study. (Received November 5, 1937.)

3. F. D. Rigby: *Note on the axioms for Boolean algebras.*

Let $E=[0, 1]$ and let $f(x, y)$ be any one of the sixteen functions in EE to E. If the Sheffer stroke operation x/y is defined by $f(x, y)$, then the axioms of Boolean algebras hold in only two cases. A systematic study of the axioms which fail to hold is made with reference to their independence and interrelations. While the axioms of Sheffer can be proved completely independent by using algebras of two elements, this is not the case for the axioms of Huntington's fourth set. These can be shown to be completely independent by aid of algebras of three or more distinct elements. (Received November 10, 1937.)

4. C. E. Sealander: *A particular third order irregular boundary value problem.* Preliminary report.

The differential system $u'''+p(x)u'+[\rho^3+q(x)]u=0$, $u(0)=u'(0)=u(\pi)=0$, is considered, $q(x)$ being a function with power series expansion in powers of x^3 only, and $p(x)$ being x times such a function. If $u_n(x)$ denotes the solutions of the system, and if

the series $\sum a_n u_n(x)$ converges uniformly for real values of x in the interval $(0, a)$, $a<\pi$, then the series converges uniformly, for complex values of x, within an equilateral triangle with center at the origin and one vertex on the axis of reals. It is shown that the series converges uniformly to a function $f(x)=x^2\phi(x^3)$, ϕ having a power series expansion in powers of x^3 only. Finally it is shown that the formal series expansion of $f(x)$ converges uniformly to $f(x)$ everywhere in the interval $(0, a)$ when suitable requirements of analyticity and continuity are imposed on the functions involved. (Received November 1, 1937.)

5. Gerhard Tintner: *A dynamic theory of duopoly.*

Duopoly exists if there are only two sellers in the market. Following the work of G. C. Evans (*Mathematical Introduction to Economics*, pp. 22 ff.) and C. F. Roos (*A dynamical theory of economics*, Journal of Political Economy, vol. 35 (1927), pp. 632 ff.; *A mathematical theory of competition*, American Journal of Mathematics, vol. 47 (1925), pp. 163 ff.; *Generalized Lagrange problems in the calculus of variations*, Transactions of this Society, vol. 30 (1928), pp. 360 ff.) the problem can be stated in a somewhat more general form: Let one duopolist maximize his profit in the interval $(0, n)$, $P=\int_0^n [p(t)x(t)-C(t)]dt$, where p is the market price, x his production, C his costs. The following relations exist: $x(t)+g[x]=f[p]$, where g is the production of his rival and a functional of his own production $x(s)$; and f is the demand, a functional of the price $p(s)$. Further, $C(t)=h[x]$, where h is a functional of $x(s)$, $0\leqq s\leqq t$. The necessary conditions for a maximum are $x(\alpha)=\int_\alpha^n l(\beta)f'[p;\alpha]d\beta$ and $p(\alpha)+l(\alpha)=\int_\alpha^n\{h'[x;\alpha]-l(\beta)g'[x;\alpha]\}d\beta$. Here l is a Lagrange multiplier (see H. Hahn, *Ueber die Lagrangesche Multiplikatorenmethode*, Sitzungsberichte Akademie der Wissenschaften, Wien, Abstrakt IIa, vol. 131 (1922), pp. 531 ff.). Here f' is the functional derivative at the point α, as are h' and g', while $p(\gamma)$ and $x(\gamma)$ are defined for $0\leqq\gamma\leqq\beta$; finally, $0\leqq\alpha\leqq n$. (Received October 30, 1937.)

6. A. A. Albert: *Quadratic null forms over a function field.*

Let L be a finite field, y an indeterminate over L, $F=L(y)$, K be algebraic of finite degree over F. Quadratic forms over K are considered and it is shown that every form in five or more variables is a null form. The proof uses the splitting field theory of E. Witt (Mathematische Annalen, vol. 110 (1934)), except in the case where L has characteristic two. Here the theorem above is proved valid for L any perfect field. In fact let L be perfect of characteristic p, $q=p^s$. Then it is shown that all forms $a_1x_1^q+\cdots+a_mx_m^q$ over K are null forms if $m>q$, but there exist non-null forms of this type for every $m\leqq q$. This result applied for $q=p=2$ is seen to give the main theorem. (Received November 11, 1937.)

7. E. T. Bell: *Polynomial approximations for elliptic functions.*

The method of a previous paper (Transactions of this Society, vol. 36 (1934), pp. 841–852) is applied to obtain successive approximations, with respect to powers of the modulus, of the elliptic functions. (Received November 5, 1937.)

8. G. A. Bliss, M. F. Smiley, and Leonidas Alaoglu: *Fields for multiple integrals of the calculus of variations.*

Two types of fields for multiple integrals of the calculus of variations have been described in recent papers by Carathéodory and Weyl. In the present paper the authors exhibit the most general type of multiple integrals which are independent of the path, and show that the fields associated with them are more general than those of

Carathéodory and Weyl. Special attention is given to the case when there are two independent variables and two dependent functions, a case which has been previously discussed in quite different fashion by Hadamard and Hilbert. (Received November 22, 1937.)

9. J. W. Calkin: *Second order partial differential operators of elliptic type. Closure.*

Let E be an open set in $(x_1, \cdots, x_n)$-space. Let p and q be real-valued Lebesgue-measurable functions defined on E, $p>0$, q and p_{kk}, $(k=1, 2, \cdots, n)$, bounded on every closed set interior to E. Let $\mathfrak{D}$ be the subset of the complex Hilbert space $\mathfrak{L}_2(E)$, each of whose elements f contains a function f^0 with the following properties: on almost all parallels to the x_k-axis which intersect E, f^0 and f_k^0 are absolutely continuous on every closed interval interior to E, $(k=1, 2, \cdots, n)$; on every bounded closed set S interior to E the Lebesgue integrals $\int_S |f_k^0|^2 dS$, $\int_S |f_{kk}^0|^2 dS$ exist, $(k=1, 2, \cdots, n)$; the Lebesgue integral $\int_E |L(f^0)|^2 dE$ exists, where $L(f^0) = -\sum_{k=1}^{n}(pf_k^0)_k + qf^0$. Let T be the transformation with domain $\mathfrak{D}$ which takes f into the element of $\mathfrak{L}_2(E)$ containing $L(f^0)$. Then $\mathfrak{D}$ is dense in $\mathfrak{L}_2(E)$, $T \equiv \tilde{T}$, $T \geqq T^*$. The condition which characterizes those elements g of $\mathfrak{D}$ which belong to the domain of T^* is a restriction on the behavior of g^e in the "neighborhood" of the boundary of E. These results are fundamental for the author's contemplated extension of the theory described in abstracts 43-3-114, 43-3-209, 43-9-323, to the case of n variables, $n \geqq 3$. (Received November 24, 1937.)

10. B. H. Camp: *Note on the probability distribution of a product.*

Two transformations are presented which are specially useful for this sort of distribution, the one producing a distribution theorem for a product corresponding to any given distribution theorem for a sum, and vice versa, the other producing the volume of a differential element in N dimensions by means of a transformation from a simpler differential volume. Some useful probability distributions of the product, which contain some known distributions as special cases, are thus derived. (Received November 1, 1937.)

11. R. V. Churchill: *Concerning Laplace transforms of periodic functions.*

Let $F(t)$ be a bounded integrable periodic function with period a. Its Laplace transform $f(z)$, together with its analytic continuation, is analytic in every finite region except possibly at the points $2n\pi i/a$, $(n=0, \pm 1, \pm 2, \cdots)$, which are simple poles with residues C_n, the Fourier constants of $F(t)$, if $C_n \neq 0$, and regular points if $C_n=0$; C_{-n} is the conjugate of C_n; also $|x| \cdot |f(x+iy)|$ is bounded in the half-planes $x \geqq b$ and $x \leqq -b$ for every $b>0$. Let $F(t)$ and its first derivative be piecewise continuous. Then $f(z)$ satisfies the additional conditions that its complex inversion integral converges for $t>0$, and that $|y_n| \cdot |f(x+iy_n)|$, where $y_n=(2n+1)\pi i/a$, is bounded for all x and n. Moreover, all of the above conditions which pertain to $f(z)$ alone form a set sufficient to insure the periodicity with period a and boundedness of the inverse transform $F(t)$, for $t>0$. Both the form and validity of the Fourier series representation of $F(t)$ under either of the above sets of conditions, on $F(t)$ or its transform, follow as a special case of a known series inversion of the Laplace transformation. (Received November 23, 1937.)

12. J. M. Clarkson: *A line transformation related to reciprocal polars of a quadric surface.*

Through an arbitrary line t of space pass two planes tangent to a fixed quadric Q at points R, S, respectively. From a fixed point O not on Q the points R, S are projected into points R', S', respectively, on Q. The tangent planes to Q at $R'S'$ meet in a line t', which is the transform of t by the line transformation. The singular lines and the invariant lines of the transformation are discussed. (Received November 18, 1937.)

13. A. H. Copeland: *A general existence theorem for simultaneous equations.*

Sufficient conditions for the simultaneous vanishing of n functions of n variables are developed in this paper. It is assumed that the functions possess first partial derivatives and that the Jacobian is non-vanishing throughout a given region R. Let g denote half the sum of the squares of the n functions. It is also assumed that a number c exists such that the region for which $g<c$ is non-vacuous and has no boundary points in common with the boundary of R. These conditions are sufficient, but probably not necessary, to insure the simultaneous vanishing of the functions. As an example it is shown that all the conditions are satisfied by a system of linear equations for which the determinant of the coefficients is not zero. As a second example an extremely simple proof of the implicit function theorem is obtained. In fact this entire paper is shorter and simpler than the treatment of the implicit function theorem alone, as it usually appears in texts on the advanced calculus. (Received November 23, 1937.)

14. H. B. Curry: *On the reduction of Gentzen's calculus LJ.*

In 1934 (Mathematische Zeitschrift, vol. 39, pp. 176–210, 405–431) Gentzen set up a "calculus LJ" and proved its equivalence to a form of the Heyting calculus which he called the "calculus LHJ." This note gives an alternative reduction of LJ to LHJ. The advantages of the new method are these: (1) it makes use of known general theorems about the Heyting calculus; (2) it shows that the derivation of any scheme in LJ can be made on the basis of only those axioms of LHJ which concern implication, together with those which actually occur in the scheme being derived. This, in combination with Gentzen's "Hauptsatz," gives a new proof of the fact that a similar separation applies to all the theorems of the Heyting calculus. (Received November 20, 1937.)

15. H. T. Davis and H. E. Jones: *Some results in the theory of random series.*

By a random series $\{y(s)\}$ is meant a series of N items, such that the autocorrelation function is zero for all lags, that is $\sum_{s=1}^{N} y(s)y(s+t)=0$, $t\neq 0$. This paper studies the properties of linear combinations of the items of random series by means of the lag-correlation functions between the series and their differences and between the differences of different orders. The results obtained are extended to continuous variables through the solution of a difference equation due to Yule. Applications to the theory of economic time series and to other statistical data are indicated. (Received November 23, 1937.)

16. J. J. De Cicco: *The inverse of Meusnier's theorem in the geometry of element series.*

This is a continuation of the paper, *The differential geometry of element series*, which was presented to this Society in March, 1937. The problem is to find the most general system of series with the Meusnier property. (For the analogous problem in ordinary space, see *The inverse of Meusnier's theorem* by Edward Kasner, this Bulletin, vol. 14 (1908), pp. 461–465). This deals with the curvature of the series of the system which have a common element and a common tangent turbine, and may be stated in three equivalent ways: 1. The reciprocal of the curvature varies as the sine of half the angle between the osculating flat field and a fixed flat field. 2. The osculating limaçon series generate a spherical field. 3. The centers of the central turbines of the osculating limaçon series form a straight line. The most general system of series with the Meusnier property is $Av''+Bw''+C=0$ where A, B, C are functions of u, v, w, v', w'. In connection with this system, there is derived a theory of *most-turbinal series* which includes, as a special case, the ∞^2 geodesic series of a field. (Received November 22, 1937.)

17. W. L. Duren: *Contractible problems of Bolza.* II.

Contractible problems of Bolza as described in a previous paper (abstract 42-11-418) did not include problems of isoperimetric type. An extension of the method described there has been found to lead to an analog of the fundamental sufficiency theorem in contracted fields of a more general type which are suitable for isoperimetric problems. (Received November 23, 1937.)

18. Churchill Eisenhart: *The power function of the χ^2-test.*

If we have N' values of a random variate classified into n' cells for which the probabilities are p_r^0, $(r=1, 2, \cdots, n')$, on the null hypothesis, H_0, and if we reject H_0 whenever $\chi^2 \geqq \chi_0^2$, then $P\{\chi^2 \geqq \chi_0^2 | H_0\}$ gives the probability of rejecting H_0 falsely with the χ^2-test. When N' is large this probability is obtainable (approximately) from the limiting distribution of χ^2 found by Karl Pearson. If, on the other hand, an alternative hypothesis, $H \neq H_0$, specifying the probabilities p_r, $(p_r \neq p_r^0$ for all $r)$, is true, then $P\{\chi^2 \geqq \chi_0^2 | H\}$ for this same $\chi_1^2 = \chi^2(H_0)$ gives the probability of rejecting H_0 when H is true, that is, the probability of detecting H with the χ^2-test. By using an artifice the writer has found that, for N' sufficiently large, $p(\chi^2 | H)d\chi^2$ is given (approximately) by R. A. Fisher's B^2-distribution with the substitutions $B^2 = \chi^2$ and $\beta^2 = \sum_r \alpha_r^2/p_r^0$, where the α_r are defined by $p_r = p_r^0 + \alpha_r/(N')^{1/2}$, from which $P\{\chi^2 \geqq \chi_0^2 | H\}$ is obtained by integrating from χ_0^2 to infinity, the integral considered as a function of χ_0^2 being the power function of the test. (Received November 18, 1937.)

19. W. W. Flexner: *Complexes whose vertices have manifolds as linked complexes.* Preliminary report.

Let an n-sphere be a generalized n-manifold of order null and define a generalized n-manifold of order p as a connected simplicial n-complex such that the linked complex of each of its vertices is a generalized $(n-1)$-manifold of order always less than p and, once at least, equal to $p-1$. It is shown that the join of two such manifolds is a generalized manifold whose order is given in terms of orders and dimensions of its components. The regular subdivision, K', of a generalized manifold K of order p is then a generalized manifold such that for $q > p-2$ the linked complexes of the ver-

tices of K' on q-simplexes of K are spheres. So the generalized manifold divides into two complexes, one a relative n-manifold (Lefschetz), the other composed of the neighborhoods of simplexes whose linked complexes are not spheres. For order 2, the complex which these have in common is the sum, S, of those linked complexes which are not spheres. The p and $n-p$ Betti numbers of the orientable generalized manifold of order 2 are connected by a relation involving also the p Betti number of S and the seam-homology numbers (Alexandroff-Hopf, p. 289) of S. (Received November 23, 1937.)

20. M. M. Flood: *Strict equivalence of matric pencils.* Preliminary report.

A rational canonical form for an arbitrary matric pencil is obtained without a preliminary transformation of indeterminates. This leads to a simultaneous classification of singular and non-singular matric pencils. The conjunctive equivalence of hermitian pencils and the strict equivalence of matric nets are also considered. (Received November 26, 1937.)

21. Lois W. Griffiths: *On hypergroups, multigroups, and product systems.*

A product system is a system of classes which is closed with respect to an associative and commutative addition process, and which is closed with respect to an associative multiplication process which is distributive with respect to the addition process. A division system is a product system for which either division, or the existence of elements analogous to the identity and inverse elements of group theory, is postulated. A division system is a generalization of the abstract system group, and of hypergroup as defined by Marty (Comptes Rendus, vol. 201 (1935), pp. 636–638). It is proved that the abstract systems, multigroup as defined by Ore (Duke Mathematical Journal, vol. 3 (1937), pp. 149–174), and hypergroup as defined by Wall (American Journal of Mathematics, vol. 59 (1937), pp. 77–98), are types of Marty hypergroups. There is defined an algebra associated with a product system, and, in particular, a hypergroup algebra. Properties of elements of the hypergroup algebra give, in particular, properties of the elements of the hypergroup. (Received November 20, 1937.)

22. D. C. Harkin: *On secant and tangent coefficients.*

Taylor-Maclaurin expansions of elliptic functions for unit modulus involve as coefficients the secant (Euler) and tangent coefficients. The theory of elliptic functions thus suggests a rapid method for the computation of these numbers. The results of Scherk (correcting Euler) are thus checked and extended. (Received November 24, 1937.)

23. Olaf Helmer: *The syntax of a language with infinite expressions.*

All languages which have so far been made the subject of syntactical investigation have had one feature in common; their expressions consisted of a finite number of symbols. In this paper the syntax of a language with expressions of infinite length is dealt with. Various possibilities as to the kind of infinite expressions that might be introduced suggest themselves, for example, infinite logical sums and products, or predicates with infinitely many argument places. The author builds up a language, which is modeled on the example of Carnap's languages I and II but which admits

of numerical expressions of infinite length, namely, of infinite dual fractions. Like language I, this language contains a K-operator, and it does not go beyond the elementary functional calculus. The requirement of definiteness, on the other hand, is given up altogether. After enumerating the symbols and laying down the formation rules, a set of axioms and a number of rules of deduction are given. The next step is the arithmetization of the syntax of this language. Since this can be carried out within the elementary theory of real numbers, of which the language itself is a representation, one can formulate the syntax of the language in itself. The remainder of the paper is concerned with the question as to the extent to which Gödel's results can be transferred to this language. (Received November 20, 1937.)

24. Paul Henle: *On the categoricity of the postulates for Boolean algebra.*

This paper constitutes a proof that, if to any one of the usual sets of postulates for Boolean algebra a postulate be added specifying the number of elements in the system as some definite finite number, then the resulting set of postulates is categorical. The proof begins by exhausting all possibilities in the case of an algebra of two elements, and then proceeds by induction on powers of two. Extensions of the theorem to algebras of an infinite number of elements are suggested. (Received November 23, 1937.)

25. T. R. Hollcroft: *The maximum number of contacts of two algebraic surfaces.*

Given two algebraic surfaces M and N, of respective orders μ and ν, whose intersection curve consists of α components of orders n_i and genera p_i, the maximum number K of contacts of M and N is proved to be $\mu\nu(\mu+\nu-4)/2+\alpha$. When $\alpha>1$, M and N have a contact at each of the Q intersections of the components and a number P of additional possible contacts. Here Q and P are both functions of n_i and p_i as well as of μ and ν, but $K=Q+P$ is a function of μ, ν, and α, only; that is, the maximum number of contacts is independent of the orders and genera of the components. Expressions for the maximum number of contacts are obtained also when common points A_i of M and N are of multiplicities r_i on M and s_i on N. Two cases are treated, A_i on distinct components and A_i on pairs of components. Numerical applications are made to quadrics and other surfaces. The maximum number of contacts can be attained by surfaces of low order. The maximum number of stationary contacts of M and N depends on the solution of the problem of finding the maximum number of cusps of a plane curve of given order and genus. This problem has not been completely solved. The maximum number of stationary contacts of M and N, however, decreases as α decreases. (Received November 26, 1937.)

26. Harold Hotelling and L. R. Frankel: *The transformation of statistics to simplify their distributions.*

When a statistic has a non-normal distribution that approaches normality for large samples, there are advantages in the use of a function of the statistic having a more nearly or exactly normal distribution. These advantages consist partly in the decreased use of tables required in tests of significance, and partly in the possibilities of further statistical studies based on a combination with other similar functions, as when one fits a regression equation to a number of correlation coefficients pertaining to various time intervals. A statistic calculated from statistics as if they were

direct observations is, as a rule, best interpreted if the statistics of which it is a function have been normalized in the manner indicated. Similarly, it is advantageous to transform a statistic whose ultimate distribution is that of χ to one having exactly, or more nearly, this limiting distribution. These transformations are illustrated in this paper by the asymptotic expansion of a normalized variate in terms of the Student t, and by that of a variate having the χ distribution in terms of the generalized Student statistic T introduced in 1931 as a substitute for "coefficients of racial likeness." From tables it appears that the first few terms of the expansions provide approximations satisfactory for many purposes. (Received November 15, 1937.)

27. Edward Kasner and J. J. De Cicco: *The classification of element transformations by means of isogonal and equi-tangential series.*

This paper is a continuation of the paper, *The geometry of isogonal and equi-tangential series*, by Kasner, Transactions of this Society, vol. 42 (1937), pp. 94–106. In this paper, the authors give a classification of all the element transformations with respect to the precise number of unions or isogonal series (equi-tangential series) which are converted into isogonal series (equi-tangential series). The main results are: 1. All the element transformations are classified into four mutually exclusive non-null classes with respect to the number of isogonal series which become isogonal series. The classes 1^0, 2^0, 3^0, 4^0 are those which transform, respectively, all ∞^∞, $2\infty^2$, ∞^2, ∞^4 isogonal series into isogonal series. 2. All the element transformations are classified into five mutually exclusive non-null classes with respect to the number of equi-tangential series which become equi-tangential series. The classes 1^0, 2^0, 3^0, 4^0, 5^0 are those which transform, respectively, all ∞^∞, ∞^2, $2\infty^2$, ∞^3, ∞^4 equi-tangential series into equi-tangential series. (Received November 22, 1937.)

28. Edward Kasner and J. J. De Cicco: *The geometry of the whirl-motion group G_6. Elementary invariants.*

This paper is a continuation of the paper, *The group of turns and slides and the geometry of turbines* by Kasner, American Journal of Mathematics, vol. 33 (1911), pp. 193–202. A turn T_α turns each element of the plane about its point through a constant angle α. A slide S_k slides each element along its line a constant distance k. These transformations generate a three parameter group which is called the whirl group. The whirl group and the rigid motion group together generate a six parameter group G_6 which is termed the whirl-motion group. It is the purpose of the authors to study the geometry of G_6. A turbine is the ∞^1 elements which are obtained by applying a turn T_α to the elements of a circle. A flat field consists of the ∞^2 elements which are co-circular with a fixed element. Under G_6, turbines and flat fields are converted into turbines and flat fields respectively. The elementary invariants between elements, turbines, and flat fields are obtained. (Received November 22, 1937.)

29. O. E. Lancaster: *Non-linear algebraic difference equations with formal solutions of the same type as the formal solutions of linear homogeneous difference equations.*

The investigations of this paper treat the following problems: Given a non-linear algebraic difference equation (1) $F(x, y(x), y(x+1), \cdots, y(x+n))=0$, where F is a polynomial with rational coefficients in its arguments $x, y(x), y(x+1), \cdots, y(x+m)$, what is the nature of F when the difference equation has solutions in common with a linear homogeneous algebraic difference equation? What is the nature of F when

(1) has a formal solution of the same general type as that of the formal solutions of linear homogeneous algebraic difference equations? And, how many linearly independent solutions of this type may a given equation possess? The study of the formal problem yields the following results: First, there is a large class of homogeneous difference equations such that any equation of the class has q and only q linearly independent solutions of the desired type, where q is the product of the degree and the order of the equation. Second, there are homogeneous difference equations which do not possess any solutions of the required type. Third, there are homogeneous equations which have an infinite number of linearly independent solutions of the given type. (Received November 20, 1937.)

30. C. H. Langford: *A theorem on deducibility for second-order functions.*

Take any set of postulates for a dense series which is complete with respect to first-order functions (see Lewis and Langford, *Symbolic Logic*, chapter 12) and add to this set the condition of Dedekind section. Take any matrix on the base of the set $F(\phi_1, \phi_2, \cdots, x_1, x_2, \cdots)$, where the ϕ's are functions of one variable on each of which the hypothesis of Dedekind's condition is imposed (that is, the ϕ's determine lower segments, and jointly line segments). Consider the class of all functions that can be derived by generalizing the variables in such a matrix. Each of these functions has its truth-value determined by the set in question, and the set may therefore be said to be complete with respect to statements about a finite number of line segments. In the presence of Dedekind's condition, the ϕ's can be progressively eliminated by a Skolem reduction procedure. (Received November 20, 1937.)

31. Jakob Levitzki: *On the equivalence of the nilpotent elements of a semi-simple ring.*

In the present note it is shown that each class of equivalent nilpotent elements of a semi-simple ring can be characterized by certain characteristic numbers; two nilpotent elements belong to the same class if and only if their characteristic numbers coincide. This is achieved by reducing each nilpotent element to a certain normal form. As a consequence one finds that the number of classes is finite. By applying the results to square matrices in a commutative or non-commutative field it is found that each nilpotent matrix can be transformed into a Jordan form. (Received November 6, 1937.)

32. R. G. Lubben: *Concerning upper semicontinuous collections and the decomposition of points of normal spaces.*

If S is a semicompletely normal Hausdorff space, there exist "portions" of it that may be decomposed into smaller portions and reassembled into larger portions. Among these are the points of the space, which in general are decomposable, and certain ideal elements, called "boundary points" (see abstract 43-9-344, this Bulletin). Every portion of S is decomposable into points which are "atomic" relative to S. Every collection of portions of S may be regarded as the set of all points of a space H Fréchet in which the operation of derivation of point sets is defined in terms of that operation for S. An "amalgamation point of S" is essentially "a perfectly compact portion of S." The theory of "upper semicontinuous" collections of amalgamation points gives a powerful tool for the study of upper semicontinuous collections of

point sets; gives criteria for the embedding of S in perfectly compact Hausdorff spaces, and for extending upper semicontinuous collections of point sets in S to upper semicontinuous collections in such spaces; and further, it gives methods for determining inverse decomposition spaces. The space of all atomic points of S is a perfectly compact Hausdorff space, and may be regarded as a universal inverse decomposition space for S. (Received November 22, 1937.)

33. N. H. McCoy and Deane Montgomery: *Subrings of direct sums and related topics.*

Let there be given a set of rings S_i, $(i \subset I)$, where I is an arbitrary range of indices. By the *direct sum* of the rings S_i, $(i \subset I)$, is meant the ring of all functions defined on I such that on i the functional values are in S_i. One of the leading results of the present paper is the theorem: *Every commutative ring without nilpotent elements is isomorphic to a subring of a direct sum of fields.* This is a generalization of results obtained by Stone (Transactions of this Society, vol. 40 (1936), pp. 37–111), by Köthe (Jahresbericht der Deutsche Mathematiker-Vereinigung, vol. 47 (1937), pp. 125–144) and by the present authors (Duke Mathematical Journal, vol. 3 (1937), pp. 455–459). The authors also show that an arbitrary ring is isomorphic to a subring of a direct sum of irreducible rings, and other related results. In particular, they establish some theorems concerning ideals in various types of rings. Finally, some of these results are applied to furnish characterizations of the commutative regular rings as defined by J. von Neumann (Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 707–713). (Received November 3, 1937.)

34. W. H. McEwen: *On the simultaneous approximation of a function and its derivatives by sums of Birkhoff type.*

Let $\{u_i(x)\}$ be the sequence of characteristic solutions of a given nth order linear differential system $L(u)+\lambda u=0$, $W_j(u)=0$, $(j=1, 2, \cdots, n)$, with regular boundary conditions defined on (a, b). By a sum of Birkhoff type of the order N is meant simply a linear combination $S_N(x)$ of $u_1, \cdots, u_N$. In this paper hypotheses are obtained under which a given function $f(x)$ and its first n derivatives can be uniformly approximated on $a \leqq x \leqq b$ in the form $|f^{(k)}-S_N^{(k)}| \leqq \epsilon_N \to 0$, $(k=0, 1, \cdots, n)$. (Received November 23, 1937.)

35. Saunders MacLane: *The uniqueness of the power series representation of certain fields and valuations.*

Krull has introduced generalized valuations, which are functions V on a field K to an ordered abelian group Γ, such that $V(a+b) \geqq \min(Va, Vb)$ and $V(ab)=(Va)+(Vb)$. The valuation V is of rank one and discrete if Γ consists of all lexicographically ordered linear forms $a_1\alpha_1+\cdots+a_m\alpha_m$, with integral coefficients a_i. Then V can be decomposed into m rank 1 valuations $V^{(i)}$, corresponding to a sequence of residue-class fields, $K, K', \cdots, K^{(m-1)}, K^{(m)}$. It is known that such a field K has a smallest extension $\bar{K}$ which is perfect with respect to V; that is, which is topologically complete with respect to each component rank 1 valuation $V^{(i)}$. Is this smallest perfect extension $\bar{K}$ uniquely determined, up to analytic isomorphism, by K? The present paper proves $\bar{K}$ unique if $K^{(m-1)}$ has characteristic 0, and shows by examples that $\bar{K}$ is not generally unique when $K^{(m-1)}$ has characteristic p. The construction of these examples depends on a new method for explicitly constructing the perfect ex-

tension $\bar{K}$ of a given K. This method is also used to obtain some structure theorems for p-adic fields, which are related to the known representation of certain perfect fields as power series fields. (Received November 22, 1937.)

36. H. M. MacNeille: *The completion of a Boolean ring and its application to integration.*

Consider a Boolean ring, K, of elements $a, b, c, \cdots$, in which an absolute value is defined such that: $|a| \geqq 0$, $|a| = 0$ if, and only if, $a = 0$ and $|a| = |a-ab| + |ab|$. Fundamental sequences are defined in K. All the postulated properties of K are extended to L, the set of fundamental sequences in K. Then K is imbedded in L and L is closed to further extension by this method. If K is a Boolean ring of point sets and K_λ the set of limit elements of sequences in K which converge in the point set sense, then (abstract 43-3-132) K_λ is a Boolean ring containing K as a subring. In general, K_λ is not closed to further extension by this method and the iterates of this extension generate the smallest Borel system containing K. If sequences in K that are null in the point set sense are null fundamental sequences, then the point sets obtained by iterating the λ-process can be identified with the abstract elements of L. No element of L is omitted. This identification permits the establishment of theorems about integration and measure in the abstract set L. The Boolean ring of half-open elementary figures with rational end points in euclidean n-space, volume being the absolute value, satisfies all the assumptions on K. (Received November 23, 1937.)

37. Karl Menger: *A theorem on relations and its applications to covering theorems of topology.*

The author considers three given abstract sets A, B, and $R \subseteqq A \times B$, and two given sets $\mathfrak{A}^*$ and $\mathfrak{B}^*$ whose elements are subsets of A and B, respectively, and which satisfy the following three conditions. I. If $B^* \epsilon \mathfrak{B}^*$ and with each $b \epsilon B^*$ there is associated a set $A_b \epsilon \mathfrak{A}^*$, then $\sum A_b \epsilon \mathfrak{A}^*$, where the sum is taken over all $b \epsilon \mathfrak{B}^*$. II. If $A^* \epsilon \mathfrak{A}^*$ and there exists a 1-to-1 correspondence between the sets A^* and B^* such that pairs of corresponding elements belong to R, then $B^* \epsilon \mathfrak{B}^*$. III. The vacuous set belongs to $\mathfrak{B}^*$. If $B^* \epsilon \mathfrak{B}^*$ and $b \epsilon \mathfrak{B}$, then $B^* + \{b\} \epsilon \mathfrak{B}^*$. For $A' \subseteqq A$ and $b \epsilon B$ let $A'(b)$ be the set of all $a' \epsilon A'$ such that $(a', b) \epsilon R$. Call $\mathfrak{R}$ the set of all ordered pairs (A', B'), where $A' \subseteqq A$, $B' \subseteqq B$, and for each $a' \epsilon A'$ there is at least one $b' \epsilon B'$ such that $a' \epsilon A'(b')$. It is then proved that the two following statements are equivalent: (1) If $A' \subseteqq A$, $B' \subseteqq B$, and $(A', B') \epsilon \mathfrak{R}$, then there exists $B^* \epsilon \mathfrak{B}^*$ such that $B^* \subseteqq B'$, $(A', B^*) \epsilon \mathfrak{R}$. (2) If $A' \subseteqq A$, $A' \notin \mathfrak{A}^*$, then there exists at least one element a' of A' such that $(a', b) \epsilon R$ implies $A'(b) \notin \mathfrak{A}^*$. If A is a space, B a set of subsets of A, R the set of those (a, b) for which a is an interior point of b, $\mathfrak{A}^*$ the set of all subsets of A whose power is $\leqq \mathfrak{n}$ (an infinite cardinal number), $\mathfrak{B}^*$ the set of all sets of $\mathfrak{n}$ or less subsets of A, then one obtains the theorems of Borel, Lebesgue, Young, and Lindelöf. (Received November 19, 1937.)

38. A. N. Milgram: *An existence theorem and some applications to topology.*

A class A of elements $a, a', a'', \cdots$, and a subset T of the cartesian product $A \times A = A^2$ are supposed given; T is such that (1) $(a, a) \epsilon T$ and (2) $(a, a') \epsilon T$, $(a', a'') \epsilon T$ implies $(a, a'') \epsilon T$. In addition one assumes given an arbitrary sequence B of objects $b_1, b_2, \cdots, b_i, \cdots$, and a subset R of $A \times B$ with the property that for any i, if $(a, a') \epsilon T$ and $(a, b_i) \epsilon R$, then $(a', b_i) \epsilon R$. If A' is a subset of A, the element a^* of A' is

called an extremal in A' if for each element a' of A' the relation $(a^*, a')\epsilon T$ implies $a^*=a'$. R is called a separator of A' if, for any pair of elements $a'\neq a''$ of A' such that $(a', a'')\epsilon R$, there exists an i for which $(a', b_i)\notin R$ and $(a'', b_i)\epsilon R$. The following theorem is proved: If A' is a subset of A, R a separator of A', and a_0 an element of A', there exists a sequence $a_0, a_1, \cdots, a_n, \cdots$, of elements in A' with $(a_n, a_{n+1})\epsilon T$, $(n=0, 1, 2, \cdots)$, such that whenever a^* is an element of A' and $(a_n, a^*)\epsilon T$ for every n, then a^* is an extremal. Various classical results in different branches of mathematics are special cases of this theorem, as, for example, the Brouwer reduction theorem and certain extreme value theorems in the calculus of variations. (Received November 19, 1937.)

39. Karl Menger and A. N. Milgram: *On linear subsets of metric spaces.*

A subset of a metric space is linear if it is congruent to a subset of E_1 (the euclidean line). The sum of a monotonically increasing sequence of linear subsets of a space is linear. The closure of a linear set is linear. By virtue of a certain saturation theorem, a subset S of a metric space, for any two distinct points p and q, contains a linear subset L_{pq} which is not a proper subset of any other linear subset of S containing p and q. If S is complete, then L_{pq} is complete, thus congruent to a closed subset of E_1. If in addition S is convex, then L_{pq} is convex and thus congruent to a segment, a ray, or E_1 itself. (Received November 19, 1937.)

40. W. E. Milne: *The remainder term for approximations of linear type.*

A large class of formulas, including most of the formulas for polynomial and trigonometric interpolation, formulas obtained by the method of least squares, formulas for numerical differentiation and integration, and formulas such as Hermite's formula for interpolation using both ordinates and slopes, all have the common characteristic of being *linear* with respect to the function to which they are applied. A general expression is obtained for the remainder term for formulas having this property. In a number of important cases this general remainder term is reduced to a form suitable for immediate applications to practical problems. In some instances, for example, Lagrange's interpolation formula and the Newton-Cotes quadrature formulas, the result is found to be identical with well known expressions for the remainder. In other cases some interesting forms have been obtained which are believed to be new. (Received November 13, 1937.)

41. M. G. Moore: *On expansions in series of exponential functions.*

Let $h(t)=\sum_{\mu=1}^{n} c_\mu \exp(\alpha_\mu t)$, where α_μ and c_μ are complex constants. Let P be the smallest convex polygon containing the points α_μ. Then $f(x)$ is expanded in a series of the form $\sum_{s=1}^{\infty}\sum_{k=1}^{\sigma} P_{ks}(x)\exp(t_{ks}x)$, where σ is a function of s, and $P_{ks}(x)$ is a polynomial of degree one less than the order of the zero t_{ks} of $h(t)$. The expansion, in the most general case considered, is convergent in a convex polygon contained in P. The method used is that of application of a generalization (obtained by the author) of the Fourier biorthogonality properties. The Fourier theorem, as stated for a Lebesgue summable function which is of bounded variation in certain neighborhoods, is one of the special cases of the convergence theorem which arises when the polygon P has only two sides. (Received November 18, 1937.)

42. M. G. Moore: *On expansions in solutions of differential equations.*

The author makes a generalization of the properties of biorthogonality, given in a previous paper (see abstract 44-1-41), to include the Birkhoff expansion theory. The term by term difference of the resulting formal expansion and the series of exponentials of the preceding paper is shown, under certain broad hypotheses, to converge uniformly to zero in certain regions. A discussion is made of the Green's functions and adjoint series belonging to the problem. The author applies the results of the paper to obtain properties of uniform convergence for the series of his preceding paper. (Received November 18, 1937.)

43. A. P. Morse: *A continuous function with no unilateral derivatives.*

A. S. Besicovitch (Bulletin de l'Académie des Sciences de Russie, vol. 19 (1925), pp. 527–540) has constructed an even continuous function B for which he asserts the properties $D_+B(x) < D^+B(x)$, $D_-B(x) < D^-B(x)$ at each point x of the interval $(-a, a)$. Later E. D. Pepper (Fundamenta Mathematicae, vol. 12 (1928), pp. 244–253) examined this same function. Besicovitch's construction is geometric and some readers have found his reasoning difficult to follow. S. Saks (Fundamenta Mathematicae, vol. 19 (1932), pp. 211–219) has proved that the "functions of Besicovitch" constitute a set of only first category in the space C of continuous functions. In the present paper there is exhibited a function which at no point of $(-1, 1)$ has a unilateral derivative (finite or infinite); in fact it has, in addition, the property that at each point of $(-1, 1)$ at least one derivative number on each side is infinite. Like Besicovitch the author employs the idea of associating with a function having dense intervals of constancy another such function. The method of association is arithmetic, however, and differs essentially from that used by Besicovitch. (Received November 22, 1937.)

44. S. B. Myers and N. E. Steenrod: *The group of isometries of a Finsler manifold.*

A Finsler manifold is an ordinary n-dimensional manifold of class C^4, provided with an arc-length defined by $\int F(x, x')dt$. The integrand F is to be invariant and of class C^3, positive, positively-regular, and positively-homogeneous of first degree in (x'). The class of Finsler manifolds includes the Riemannian manifolds. A Finsler manifold M becomes a metric space if distance $\rho(A, B)$ is defined as the greatest lower bound of the lengths of arcs AB. An *isometry* of M is a distance-preserving homeomorphism of M into itself. The group of isometries of M can be topologized so that a sequence of isometries T_i converges to an isometry T if and only if $T_i(P)$ converges to $T(P)$ for every P on M. The object of the present paper is to prove that under this topology the group of isometries of a compact Finsler manifold is an abstract Lie group. (Received November 23, 1937.)

45. C. J. Nesbitt: *Relations between the coefficients of the modular representations of groups.*

In the theory of ordinary representations of finite groups, the well known orthogonal relations are fundamental. Extensions of these relations hold for modular representations. Let a prime ideal divisor of a rational prime p be the modulus. Denote by

U a modular indecomposable constituent of the regular representation R of a finite group G; by F, the modular irreducible representation of G which is uniquely associated with U; by f, the degree of F; and by A, the square matrix, of degree f, appearing in the lower left corner of U. If, for the element g of G, $A(g)=(a_{\kappa\lambda}(g))$, $F(g)=(f_{\kappa\lambda}(g))$, it results that (1) $\sum_{\sigma\subset G} a_{\alpha\rho}(g^{-1})f_{\sigma\beta}(g)=c\delta_{\alpha\beta}\cdot\delta_{\rho\sigma}$, where $c\not\equiv 0 \pmod{p}$; (2) $\sum_{\sigma\subset G} b_{\alpha\rho}(g^{-1})f_{\sigma\beta}(g)=0$, where $b_{\alpha\rho}$ is a coefficient of some indecomposable constituent of R, but is not contained in A. By a related argument it follows that there is no indecomposable representation, other than U itself, which contains U as constituent. U is said to be of the first kind if each of its irreducible constituents is equivalent to F. For such U the trace of A is a class function, and a character relation results from (1). (Received November 23, 1937.)

46. Rufus Oldenburger: *Non-singular linear combinations of general forms.*

Each form $M=a_{ij\ldots m}x_iy_j\cdots z_m=x_iM_i$, $(i=1,\cdots,r)$, can be written in a field F as the sum $S=A_\alpha B_\alpha\cdots D_\alpha$, $(\alpha=1,\cdots,s)$, for some s, where $A_\alpha,\cdots,D_\alpha$ involve the variables $x_i,\cdots,z_m$, respectively. A form is said to be non-singular in F if it has a representation S where the forms in each of the sets $(A_\alpha),\cdots,(D_\alpha)$ are linearly independent. Non-singular forms were studied elsewhere by the author (Transactions of this Society, 1936); it was proved that for each non-singular form M there exist values a_i of x_i in F such that a_iM_i is non-singular. If the forms in each of the sets $(B_\alpha),\cdots,(D_\alpha)$ are linearly independent, M is said to be non-singular in F on $j,\cdots,m$. The principal contribution of this paper is a proof that if M is non-singular in F on $j,\cdots,m$, there exist values a_i of x_i in F such that a_iM_i is non-singular. Using this result, one can prove that a form M of degree $\geqq 4$ is non-singular in F on $j,\cdots,m$ if and only if (a) there exist values a_i of x_i in F such that $H=a_iM_i$ is non-singular; (b) if (a) is satisfied, the x's being so labeled that $a_1\not\equiv 0$ in H, the non-singular linear transformations which bring H into the canonical form $x_1\cdots z_1+\cdots+x_n\cdots z_n$ bring $M_2,\cdots,M_r$ into forms with diagonal matrices. (Received November 20, 1937.)

47. E. W. Paxson: *On the postulates for linear topological spaces.*

Von Neumann has given a useful set of postulates for linear topological spaces (Transactions of this Society, vol. 37 (1935), p. 4). The author shows by simple examples that (2), (4), (5), and (6) (loc. cit., Definition 2b) constitute an independent set, demonstrating that, primarily on the basis of the continuity postulate (4), the Hausdorff intersection axiom (3) is fulfilled. (Received December 3, 1937.)

48. G. B. Price: *The theory of integration.*

This paper considers the integration of functions defined over certain abstract spaces with values in Banach spaces. It extends and unifies the theories of integration developed by Bochner, Garrett Birkhoff, and others, and determines the interrelations and limitations of these various special theories. In order to establish these results, a class of sets is defined and studied which includes convex sets as a special case. The convergence of infinite series whose terms are sets is investigated. Measurable functions are defined, their properties established, and their integrals investigated; the measurable functions of Bochner (Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276) are a special case of the measurable functions here defined. The Riemann-Stieltjes integral is treated, and the Darboux theory is extended to these

integrals. Finally, some results are given on singular integrals and Fourier series. (Received November 20, 1937.)

49. G. Y. Rainich: *Dirac equations and conditional invariants.*

Special cases of Dirac equations have been translated into ordinary tensor analysis by Gordon Fuller (abstract 42-11-445), who used certain quadratic combinations of the psi's introduced by Darwin. In connection with some of these special cases, the author introduced conditional invariants (abstract 43-3-180). Early this year, Whittaker introduced another set of quadratic quantities and expressed the Dirac equations in terms of them, using an irrational differential operation. In the present paper, quantities are introduced which are closely related to Whittaker's, but avoid the use of both self-dual six-vectors and the complex conjugate. In terms of these quantities, the Dirac equations are written using ordinary tensor analysis without irrational operations, but employing conditional invariants. In three-dimensional terminology the situation may be described as follows: consider three mutually perpendicular vectors of equal length; two of them may be interpreted as electric and magnetic vectors, the third as the momentum vector of matter, their common length as density; then the left sides of the equations are the left sides of Maxwell's equations but the right sides, instead of being zero, are components of a certain pair of singular conditional covariant vectors. (Received November 23, 1937.)

50. W. C. Randels: *On the absolute summability of Fourier series.*

An example is given of a function whose Fourier series is absolutely summable $|A|$ but not $|C_\alpha|$ for any α. (Received November 3, 1937.)

51. W. T. Reid: *A theorem on quadratic forms.*

In this note, the following theorem is proved: If $A[x]$ and $B[x]$ are real quadratic forms in (x_α), $(\alpha=1, \cdots, n)$, and $A[x]>0$ for all real $(x_\alpha)\neq(0_\alpha)$ satisfying $B[x]=0$, then there exists a constant λ_0 such that $A[x]-\lambda_0 B[x]$ is a positive definite form. This result is of use in considering the Clebsch condition for multiple integrals of the calculus of variations. Recently, A. A. Albert (see abstract 43-11-395) has obtained an algebraic proof of this result. The proof of this note is more analytic in character than that of Albert, and is of interest because in the proof itself there is obtained directly, in terms of the roots of the associated characteristic equation, the interval in which the value λ_0 of the theorem must be chosen. (Received November 23, 1937.)

52. R. F. Rinehart: *Commutative algebras which are polynomial algebras.*

By a polynomial algebra P modulo $p(x)$ is meant the algebra of the residue classes of the ring of all polynomials with coefficients in a field F, modulo the polynomial $p(x)$ with coefficients in F. If $p(x)$ is irreducible, then P is a field. In this paper a study is made of the case where $p(x)$ is not necessarily irreducible. If $p(x)$ is reducible, then P is a direct sum of primary polynomial algebras whose moduli are the powers of the distinct irreducible factors of $p(x)$, and, conversely, a direct sum of polynomial algebras is a polynomial algebra. This fact is used to determine the structure of commutative algebras which are equivalent to polynomial algebras. (Received November 23, 1937.)

53. W. C. Risselman: *On periodic continued fractions.*

Infinite periodic continued fractions of the Stieltjes-Grommer type are under consideration. Trigonometric forms for the numerators and denominators of the convergents are used to study the sequence of convergents on the intervals of the real axis on which the discriminant of the associated quadratic equation is non-positive. (Received November 26, 1937.)

54. J. B. Rosser: *On the consistency of Quine's "New foundations for mathematical logic."*

The author has investigated the consistency of the system of logic presented by Quine in *New foundations for mathematical logic* (American Mathematical Monthly, vol. 44 (1937), pp. 70–80). This was reviewed by Bernays in the Journal of Symbolic Logic, vol. 2 (1937), pp. 86–87. As Bernays pointed out in his review, the system is inconsistent if one of Quine's definitions of stratification be used. Using the other definition of stratification, an attempt has been made to obtain the usual contradictions, but without success. The reasons for the failure of the particular attempts are explained. In attempting to find ω-inconsistencies or inconsistencies of a more subtle character, an inferential rule of a type due to Kleene was added to the system. This rule has the property that the system will not contain any inconsistencies after addition of the rule unless it contained inconsistencies (possibly of a very indirect sort) before addition of the rule. This rule makes possible the proof of a number of interesting results, among them the axiom of infinity, but apparently gives no inconsistencies. (Received November 20, 1937.)

55. J. B. Rosser and R. J. Walker: *On the transformation group for diabolic magic squares of order four.*

A group of transformations is given by which any diabolic magic square of order four is transformed into another such square. Furthermore every such square can be derived from a single normalized square by a unique transformation of the group. Generators of the group are given and the group is shown to be of order 384. Hence there are exactly 384 diabolic magic squares of order four. (Received November 17, 1937.)

56. E. H. Rothe: *Theory of the topological order in linear topological spaces.*

As was shown in a former paper to appear in Compositio Mathematica, the topological theory of the order of a point with respect to the image of a sphere holds in Banach space. It is now shown that the theory holds also in certain non-metric spaces. (Received November 22, 1937.)

57. O. F. G. Schilling: *Algebraic theory of abelian functions.*

Let K be a function field of genus p whose field of constants is an arbitrary, algebraically closed field. The author generalizes the classical theory of abelian functions. He considers the algebraic variety of all unordered p-tuples of points on K and develops on this manifold the analogue of the classical theory. Using van der Waerden's notion of generic point one sees that the rational functions of this variety form a field A of p variables which is given as the subfield of the p-fold direct product of

K with itself. By applying the theory of algebraic correspondences, it is proved that A and its associated variety admit the group G of all divisor classes of degree 0 in K as a group of birational transformations. The group of all birational transformations on A contains the reflection. The group G_n of all divisor classes of order n is a finite group for each n; G_n is the Galois group of A over the field A_n of natural multiplications of A by n; and A is unramified over A_n. There exists also an equivalent for the complex multiplications. All these results will be published in a series of papers, partially in collaboration with H. Busemann. (Received November 19, 1937.)

58. O. F. G. Schilling: *A note on infinite perfect fields.*

Let k be a field which is perfect with respect to a continuous valuation of rank one and let κ be its field of residue classes. If k' denotes a subfield of k which is dense in k, in the sense of the given valuation, and whose field of residue classes coincides with κ, then it is shown that each normal extension $k(A)$ of k can be obtained as the join of a suitable superfield $k'(A')$ of k' by means of k: $k(A)=k'(A')k=k(A')$. This theorem is of importance for the investigation of the structural theory of general infinite perfect fields. (Received November 12, 1937.)

59. O. F. G. Schilling: *On the structure of local class field theory.*

Let k be a field which is perfect with respect to a discrete valuation of rank one and whose field of residue classes κ is a finite Galois field. The theory of finite abelian extensions and normal algebras over k was developed by H. Hasse. The author investigates problems of the following type: given a perfect field k and a theorem of Hasse's theory as axiom for k, what can be said about the structure of k? It is shown that the more significant theorems taken as axioms for k imply that, for each integer n, there exists exactly one cyclic unramified extension of k. In order to obtain the most general p-adic fields, and x-adic fields, where the residue fields κ are infinite Galois fields which admit extensions of any degree n, one has to introduce another axiom, namely, that the assumption concerning the validity of a theorem is to hold also for all perfect subfields k' of k whose residue class fields κ' are subfields of κ. This theory is in a certain sense an algebraico-arithmetical equivalent to N. Jacobson's topological theory of perfect fields. However, the class of perfect fields which is described is larger than his. (Received November 12, 1937.)

60. I. J. Schoenberg: *Metric spaces and positive definite functions.*

Let $\mathfrak{S}$ be a semi-metric space with the distance function $\overline{PP'}$. A real continuous even function $g(t)$ is said to be positive definite (p.d.) in $\mathfrak{S}$ if $\sum_1^n g(\overline{P_iP_k})\rho_i\rho_k \geqq 0$, for arbitrary real ρ_ν, any n points P_ν of $\mathfrak{S}$, and this for $n=2, 3, 4, \cdots$. Thus $g(t)=e^{-\lambda t^2}$, $(\lambda>0)$, is p.d. in Hilbert space $\mathfrak{H}$. As Theorem 1, it is proved that a semimetric space $\mathfrak{S}$ is isometrically imbeddable in $\mathfrak{H}$ if and only if $\mathfrak{S}$ is separable and all functions of the family $e^{-\lambda t^2}$, $(\lambda>0)$, are p.d. in $\mathfrak{S}$. Let $\mathfrak{S}(\gamma)$ denote the new space obtained from $\mathfrak{S}$ if the metric $\overline{PP'}$ is replaced by $(\overline{PP'})^\gamma$, $(\gamma>0)$. A new simple proof is given of the author's previous result: $\mathfrak{H}(\gamma)$, $(0<\gamma<1)$, is isometrically imbeddable in $\mathfrak{H}$. Theorem 1, together with previous results on screw lines in $\mathfrak{H}$ by von Neumann and the author, leads to the following purely analytical result in the theory of p.d. functions: The most general positive function $f(x)$ whose positive powers $[f(x)]^\lambda$, $(\lambda>0)$, are all positive definite is of the form $f(x)=\exp\{c-\int_1^\infty (\sin xu)^2/u^2 d\sigma(u)\}$, where $\sigma(u)$ is non-decreasing for $u\geqq 0$ such that $\int_1^\infty u^{-2}d\sigma(u)$ is finite, while c is a real constant. A further consequence of Theorem 1 is as follows: If $0<p\leqq 2$, then $L_p(\gamma)$, $(0<\gamma\leqq p/2)$,

is isometrically imbeddable in $\mathfrak{H}$. For $p=2$ we have the previous result on $\mathfrak{H}(\gamma)$. Similar statements for $p>2$ depend on yet unsolved problems in the theory of p.d. functions of several variables. (Received November 22, 1937.)

61. W. E. Sewell: *Jackson summation of the Faber development.*

Let C be an analytic Jordan curve in the z-plane. Let $f(z)$ be analytic in C, continuous in $\overline{C}$, and let $f^{(p)}(z)$, $p \geq 0$, satisfy a Lipschitz condition of order α, $0 < \alpha \leq 1$, on C. Then one has $\left| f(z) - \sum_{\nu=0}^{n} d_{n\nu} a_\nu P_\nu(z) \right| \leq M/n^{p+\alpha}$, z in $\overline{C}$, where M is a constant independent of n and z, $\sum_{\nu=0}^{n} a_\nu P_\nu(z)$ is the sum of the first $n+1$ terms of the development of $f(z)$ in the Faber polynomials belonging to C, and $d_{n\nu}$ is the Jackson summation coefficient. (Received November 16, 1937.)

62. H. A. Simmons: *Affine differential geometry of plane curves from the point of view of Wilczynski.*

This paper contains: certain comparisons between Blaschke's method of studying the affine differential geometry of plane curves and a corresponding method which Wilczynski suggested to the author; an affine analogue of a general theorem of E. P. Lane on the differential equation which is appropriate for use in Wilczynski's projective differential geometry of curves in a linear space of $n \geq 2$ dimensions; application of the Lie theory of differential equations to distinguish sharply between the affine and the projective differential geometries of plane curves from Wilczynski's point of view; and analogs of most of the results of Wilczynski on projective differential geometry of plane curves. Special results of interest in this paper are: equations of the osculating conic and of the osculating cubic, coordinates of the Halphen point, and a proof of the fact that in the affine differential geometry of plane curves a curve may be anharmonic when the coefficients of the associated differential equation are not all constants. (Received November 15, 1937.)

63. M. F. Smiley and L. R. Wilcox: *Metric lattices.*

In the theory of continuous geometries a real-valued dimension function $D(a)$ defined over a lattice L and satisfying $D(a+b)+D(ab)=D(a)+D(b)$, $D(c)<D(d)$ for $c<d$, is shown to give rise to a metric $\delta(a, b)=D(a+b)-D(ab)$. This topology and its relation to that determined by the partially ordering relation have been studied by J. von Neumann (Proceedings of the National Academy of Sciences, vol. 22 (1936), p. 106). In a study of affine geometry by means of the theory of lattices it is shown that dimension functions satisfy $D(a+b)+D(ab) \leq D(a)+D(b)$ (K. Menger, Annals of Mathematics, vol. 37 (1936), pp. 465–466). The present paper deals with lattices in which dimension functions satisfying conditions weaker than this latter condition are defined. A metric topology may be introduced by defining $\delta(a, b) \equiv 2D(a+b)-D(a)-D(b)$; a characterization is given for metrics arising in this manner. Certain interrelations between this topology and the order topology are investigated; a typical result is that, under several mild hypotheses, completeness of L in the metric implies that L is continuous. (Received November 22, 1937.)

64. I. S. Sokolnikoff and E. S. Sokolnikoff: *Torsion of regions bounded by circular arcs.*

The torsion problem requiring a harmonic function to assume the value $(x^2+y^2)/2$ on the boundary of some finite regions, bounded by pairs of circular arcs, is solved in

closed form. The method used is to map the regions in question upon a unit circle and evaluate some integrals equivalent to those of Schwarz. (Received November 19, 1937.)

65. Daisy M. Starkey: *On a fiducial test of the significance of the difference of the means of two normally distributed populations which are not known to have equal variances.*

Essentially the problem consists of finding the distribution of $at+bt'$, where a and b are constants, and t and t' are distributed in Student's distribution. This paper represents an attempt to generalize the recent work of R. A. Fisher concerning samples of two, and considers: (i) The case in which the sample numbers N, N' are even and small. An expression for the characteristic function of Student's distribution valid for all even values of N is used, and, if $N=N'$, the distribution and probability integral of θ are derived, where $(N-1)^{1/2}\tan\theta=(at+bt')/(|a|+|b|)$. The population means may be said to differ significantly if the quantity $\theta_1=\arctan(\bar{x}-\bar{x}')/n^{1/2}(s+s')$ is significantly large, where $s^2=\sum(x-\bar{x})^2/N(N-1)$, and $n=N-1$. (ii) The case in which N and N' are large and of the same order of magnitude. R. A. Fisher's asymptotic expansion of Student's distribution is used, and, if $N=N'$, an asymptotic expansion for the distribution of $(at+bt')/(a^2+b^2)^{1/2}$ in terms of n is derived, and also for the probability integral, in which the quantity $(\bar{x}-\bar{x}')/(s^2+s'^2)^{1/2}$ may be tested. If the ratio of the population variances were known, the exact distributions of $(\bar{x}-\bar{x}')/n^{1/2}(s+s')$ and $(\bar{x}-\bar{x}')/(s^2+s'^2)^{1/2}$ should be used. (Received November 30, 1937.)

66. Eugene Stephens: *The purely operational solution of linear differential equations with polynomial coefficients.*

By the transformation $x=e^z$ and the operational identity $x^nD^n\equiv\prod_{\alpha=0}^{n-1}(\theta-\alpha)$, equations of the form $\sum_k F_{kk}(s)\cdot D^k\cdot y=Q(x)$, with $F_{kk}=\sum_k C_k x^k$, become equations of the form $\sum_m f_m(\theta)\cdot e^{mz}\cdot y=Q(e^z)$. Division by any one of the $f_m(\theta)$ gives a form $[1+\sum_i\phi_i]\cdot y=P(z)$, the operational solution of which is $y=[1+\sum_i\phi_i]^{-1}\cdot[P(z)+0]$. A reciprocal theorem which transforms the inverse operator $[1+\sum_i\phi_i]^{-1}$ into a series of direct operators is then applied. Two reciprocal power series in x, one with the coefficients ϕ_i and the other with the undetermined coefficients F_k as the direct operators, are multiplied together and the coefficients equated to obtain the relations between the ϕ_i and the direct operators F_k. As many series solutions may be obtained as there are terms $f_m(\theta)$ in the (z,θ) form, and each of these has as many arbitrary constants as the degree in θ of the respective $f_m(\theta)$ used as a divisor. Systems are transformed in the same manner, but the eliminant of the system is obtained by using the determinant of the matrix of the (z,θ) operator coefficients as the operator on an unknown function. The individual unknowns are then proportional to the cofactors of any row in the matrix of coefficients, the function of proportionality being the solution of the eliminant equation. Regular and singular systems are considered. (Received November 23, 1937.)

67. Otto Szász: *Some inequalities in the theory of Fourier series.*

Suppose $f(x)$ is real or complex valued, periodic of period 2π, and Lebesgue integrable on $(-\pi,\pi)$. Denote its Fourier series by $f(x)\sim\sum_{-\infty}^{\infty}c_\nu e^{i\nu x}$, and let $P_n(z)=\sum_1^n \nu c_\nu z^\nu+\sum_{n+1}^{2n}(2n-\nu)c_\nu z^\nu$. It is shown that if $|f(x)|\leq\mu$ for all x, then $|P_n(z)|/n\leq\mu$, for $|z|\leq 1$, $n=1, 2, 3, \cdots$. If $f(x)$ is real, then also $|\Re P_n(e^{iz})|/n\leq 2\mu/\pi$,

$|\Im P_n(e^{iz})|/n \leqq 2\mu/\pi$, $n=1, 2, \cdots$. The cases where equality occurs in the above are also considered. The last inequality for $x=0$ leads to an improvement of an estimate given by the author in the American Journal of Mathematics, vol. 59 (1937), p. 701. (Received November 19, 1937.)

68. H. S. Thurston: *Matric conjugates in a ring $R(A)$.*

If e_i, $(i=1, 2, \cdots, n)$, are the principal idempotent elements of a square matrix A having distinct latent roots α_i, it is well known that (a) any matrix $M=f(A)$ in the ring $R(A)$ has latent roots $\mu_i=f(\alpha_i)$ and (b) $M=\sum \mu_i e_i$, an equality which may be symbolically expressed as $M=(\mu_1, \mu_2, \cdots, \mu_n)$. This symbolism provides an exceptionally convenient way of determining sets of matrices conjugate to M, of the types considered by Taber, Franklin, Sokolnikoff, and Hermann. Sets of conjugates belonging to none of these types are also obtained. (Received November 20, 1937.)

69. C. B. Tompkins (National Research Fellow): *Deformations of the inner equator of a torus.*

This paper shows that if the inner equator of a torus is continuously deformed completely around the surface and back onto itself, the maximum length necessary is that of the closed geodesic of the same topological type as the equator and intersecting the outer equator in diametrically opposite points; furthermore, there is no deformation which does not contain a curve of at least this length. This is a result suggested by Marston Morse. That the length suffices is shown by a simple example; the necessity of including a curve of at least this length is shown by proving that some curve of the family passes through diametrically opposite points of the outer equator. This last is a corollary of the proposition: If A, B, and C are concentric circles in a plane, A inside B which is inside C, any continuous deformation of A into C passes through some curve which intersects B in diametrically opposite points. (Received November 23, 1937.)

70. W. J. Trjitzinsky: *Theory of functions of a complex variable defined over general sets.*

The author develops a theory of functions $f(z)=u(x, y)+iv(x, y)$ of a variable $z=x+iy$, possessing unique first derivatives on a perfect set E. More precisely, he studies "general monogenic" functions for which u, v, the first partials of u and v, and the second partials of u are continuous in E. Here u is uniform and $\Delta u=0$ in E, (Δ is the Laplacian operator), while v is a harmonic conjugate in E of u. It is established that such functions $f(\alpha)$ are representable in E as $h(\alpha)+\iint q(x, y) \log (z-\alpha)dxdy$. Here $h(\alpha)$ is analytic and $q(x, y)$ is real continuous, $q(x, y)=0$ in $C(E)$ (the complement of E). The set $C(E)$ is covered by a set of domains $|z-A_i|<\gamma_i$, $(i=1, 2, \cdots)$. An extensive investigation is made on the basis of the principles according to which the faster $q(x, y)\to 0$ (as $z\to$the frontier of $C(E)$) and the faster $\gamma_i\to 0$ (as $i\to\infty$) the more regularity properties (existence of derivatives, various representations, various uniqueness properties) will the corresponding g.m. functions possess. The work will appear (in French) in the Annales de l'École Normale Supérieure. (Received November 22, 1937.)

71. A. W. Tucker: *Symmetric and alternating products of circles.*

The following results are established by quite elementary methods: (1) the symmetric product of $2n+1$ circles is homeomorphic with the direct topological product

of a $2n$-cell and a circle, (2) the symmetric product of $2n$ circles is homeomorphic with a $2n$-dimensional "Möbius band," and (3) the alternating product of $n+1$ circles is homeomorphic with the direct topological product of an n-sphere and a circle. (Received November 24, 1937.)

72. H. E. Vaughan: *On the characterization of abstract spaces by postulating the existence or non-existence of certain types of metrics.*

Let E be a metrisable space, M the class of all metrics consistent with the topology of E. Then M contains the following subclasses: B, C, TB, and TC, consisting of all metrics in which E is bounded, complete, totally bounded, and totally complete, respectively. Using these concepts as principles of classification, M can be expressed as the sum of seven disjoint sets, which generate a Boolean algebra with 2^7 elements, in general distinct. This paper investigates all cases in which this algebra degenerates into algebras with fewer elements, thus obtaining systematically all theorems of the type suggested in the title. (Received November 23, 1937.)

73. H. S. Wall: *Abelian hypergroups.* Preliminary report.

A set A is an abelian hypergroup if: (1) $a+b=[c]$ for $a\varepsilon A$, $b\varepsilon A$, where $[c]$ is a finite or infinite subset of A; (2) $a+b=b+a$; (3) $a+(b+c)=(a+b)+c$; there is a $0\varepsilon A$ such that if $a\varepsilon A$, $a+0=a$; (5) $a\varepsilon A$ implies that there exists uniquely $-a\varepsilon A$ such that $0\varepsilon a=a$; (6) $a+b=[c]$ implies $-a-b=[-c]$. The hypergroup A is homomorphic with A' if there exists a function f such that if $a\varepsilon A$, $f(a)\varepsilon A'$; $f(0)=0'$; and $f(a)+f(b)=[f(c)]$ where $a+b=[c]$. A "hyperfield" of these operators f has the property that multiplication is single-valued and addition is many-valued. Difference hypergroups may be formed, and if $A=A'$ there exists the isomorphism $A-f(A)\approx g(A)-g(A)\cap f(A)$, where g is any one of the "branches" of $1-f$. The separate branches of $f(a)+g(a)$ enjoy a sort of "continuity" in a certain subhypergroup containing $a\varepsilon A$. The Jordan-Hölder theorem, and the notions of linear dependence and rank can be extended to A; and topological systems of abelian hypergroups can be defined to which may be carried the formal theory of topological systems of abelian groups. (Received November 17, 1937.)

74. S. E. Warschawski: *On the degree of approximation in some convergence theorems in conformal mapping.*

Let $w=f(z)$ map the circle $|z|<1$ conformally onto the interior R of a closed Jordan curve C, $(f(0)=w_0, w_0$ in R, and $f'(0)>0)$. It is known that, under a suitable continuous deformation of C, $f(z)$ and $f'(z)$ vary continuously with C. In the present paper estimates for the degree of the variation of $f(z)$ and $f'(z)$ are obtained. (1) Suppose: (a) C lies in the annular region $1-\epsilon\leqq|w|\leqq 1+\epsilon$, and (b) C has continuously turning tangent $\theta(s)$, ($\theta(s)$ = inclination angle, s = arc length), and $|\Delta\theta/\Delta s|\leqq 1+\epsilon$. If $w_0=f(0)=0$, $f'(0)>0$, then there is an absolute constant M such that, in $|z|\leqq 1$, $|f'(z)-1|\leqq M\epsilon$. (2) Suppose: (a) C and C^* are two closed Jordan curves such that C^* lies in the region G_ϵ, obtained when a circle of radius ϵ is described about every point of C; (b) C and C^* have continuous curvature $\kappa(s)$ and $\kappa^*(s^*)$ respectively, and, if P^* is a point on C^* within the circle of radius ϵ about P of C, then $|d\kappa/ds(P)-d\kappa^*/ds^*(P^*)|\leqq\epsilon$; (c) C and C^* lie within a circle $|w-w_0|=R$ and contain a circle $|w-w_0|=r$ in their interiors. If $f(z)$ and $f^*(z)$ map $|z|<1$ onto the interior of C and C^* respectively and if $f(0)=f^*(0)=w_0$, $f'(0)>0$, $f^*(0)>0$, then $|f(z)-f^*(z)|\leqq M\epsilon$,

where M is a constant depending only upon r and R. (3) Some weaker estimates are obtained under weaker hypotheses. (Received November 23, 1937.)

75. D. L. Webb: *A set of postulates for an algebra of n-valued logic.*

All of the elements of Boolean algebra can be expressed in terms of elements which E. V. Huntington (Transactions of this Society, vol. 5 (1904), pp. 308, 309) calls irreducible elements and B. A. Bernstein (American Journal of Mathematics, vol. 57 (1935), pp. 733–742) minimal elements. A set of independent postulates is found for the generalization of these elements that occur in the algebra of n-valued logic and for their combination by means of an operation of addition. The algebra of n-valued logic is defined as consisting of all possible sums of these irreducible elements. All algebras, for a given n, and with the same number of elements, are found to be simply isomorphic. In the case for $n=2$, the algebra becomes a Boolean algebra. (Received November 20, 1937.)

76. G. T. Whyburn: *Interior transformations on certain curves.*

Let $T(A)=B$ be an interior transformation where A is compact. In this paper the following results are established: (1) If D is any dendrite in B there exists a dendrite E in A which maps topologically onto D under T. If A is a boundary curve (that is, a locally connected continuum every true cyclic element of which is a simple closed curve), so also is B. (3) If the boundary of every connected open subset of A is totally disconnected, then T is necessarily light, that is, dim $T^{-1}(b)=0$ for every $b \in B$. (4) If A is a dendrite, then (a) for every $x \in B$, $T^{-1}(x)$ contains at most a finite number of cut points of A, and (b) if for each $x \in B$ one lets $k(x)$ be the number of points in $T^{-1}(x)$, then for any two points a and b of B (which is necessarily a dendrite) and any point x on the arc ab of B, one has $k(x) \leqq k(a)+k(b)-1$. Finally, an example is given of an interior transformation of a planar graph A onto a non-planar graph B. (Received November 17, 1937.)

77. G. T. Whyburn: *Interior transformations on 2-dimensional manifolds.*

In this paper it is shown that the image of a 2-dimensional manifold (with or without boundaries) under any interior light transformation (that is, a transformation mapping open sets into open sets and mapping no continuum into a point) is itself a 2-dimensional manifold. In particular, any interior light image of a sphere is necessarily a sphere, a projective plane, or a 2-cell. By factoring the general interior transformation into a monotone one and a light interior one, an analysis of the general interior transformation on 2-dimensional manifolds is made possible. (Received November 3, 1937.)

78. W. M. Whyburn: *On the one dimensional Green's function.*

This paper studies the Green's function for nth order linear differential systems with boundary conditions of integral type or which involve a finite or convergent infinite sequence of points of the interval. The familiar method of variation of parameters is used as a starting point in the development of the Green's matrix and in the study of its properties. The results of the paper are used as a guide to a study of new types of boundary value problems for partial differential equations. (Received November 12, 1937.)

79. L. R. Wilcox: *Modularity in the theory of lattices.*

The author has investigated lattices called *affine lattices* which arise from modular lattices by deletion of certain elements (abstract 43-3-159). The present work begins an axiomatic study of lattices possessing some of the properties of affine lattices. The rôle of the modular axiom in the usual theory of independence is studied in detail; a theory of independence for non-modular lattices is obtained by means of an investigation of the closure properties of the set of all pairs (b, c) satisfying $(a+b)c = a+bc$ for every $a \leqq c$. A well known result concerning the equivalence of ascending and descending chain conditions for complemented modular lattices is generalized to a considerably wider class of lattices. Finally, it is shown that the essential results of K. Menger (Annals of Mathematics, vol. 37 (1936), pp. 464–466) for affine geometry hold without assumption of the axiom of complementation, the methods used being an extension of those of Dedekind for modular lattices. (Received November 22, 1937.)

80. Margarete C. Wolf: *The transformation of bases of relative linear sets.*

The definitions of linear extension and linear independence relative to a matrix are those introduced in *Relative linear sets and similarity of matrices whose elements belong to a division algebra*, by M. H. Ingraham and M. C. Wolf, Transactions of this Society, vol. 42 (1937), pp. 16–31. The basic number system is a field F. If $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the linear extension, relative to the matrix M, of vectors $\xi_1, \xi_2, \cdots, \xi_k$ which are linearly independent relative to M, and if for a set of polynomials f_{ij} in F, $\eta_i = \sum_{j=1}^{k} f_{ij}(M)\xi_j$, necessary and sufficient conditions that the set η_i, $(i = 1, 2, \cdots, l)$, be a linearly independent base relative to M for $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ are obtained for the general problem from cases in which the minimum polynomial g_i such that $g_i(M)\xi_i = 0$ is g^{u_i}, a power of an irreducible polynomial g. Necessary and sufficient conditions that such η_i form a linearly independent base relative to M are: (1) for every g^{u_i} there is one and only one η_j such that g^{u_i} is minimum for $g^{u_i}(M)\eta_j = 0$; (2) the determinant of the matrix (f_{ij}) is not zero in the ring of polynomials reduced modulo g. The problem is being studied if F is a division algebra, not necessarily commutative. (Received November 19, 1937.)

81. J. M. Feld: *On certain groups of birational contact transformations.*

L. Autonne studied finite linear and quadratic groups of birational contact transformations (crémoniens) in S_2 (Journal de Mathématiques, (4), vol. 3 (1887) and vol. 4 (1888)). In this paper an infinite mixed group Γ is constructed. Γ has an invariant abelian subgroup G. The members of Γ leave a two-parameter family of triangular-symmetric curves invariant. Contragradient variables are used to represent line elements. If (x, u) and (y, v) are a pair of corresponding elements with respect to any member of Γ, the cross-ratio of u and the three lines joining x to the vertices of the fundamental triangle equals the cross-ratio of v and the lines joining y to the same vertices. The dual of this property is also possessed by the transformations. The group Γ has an analogue in S_n. (Received December 28, 1937.)

82. Aaron Fialkow: *Conformal geodesics in Riemann spaces.*

The images $\bar{G}$ in $\bar{V}_n$ of the geodesics of V_n under a conformal transformation $d\bar{s} = \lambda ds$ are called *conformal geodesics* of $\bar{V}_n$ and similarly for the curves G in V_n.

A geometric characterization of conformal geodesics in special spaces was given by Kasner and Lipke, who considered these curves from another viewpoint (dynamics). A simpler derivation of these characteristic properties which has wider validity is given. At each point of $\overline{V}_n$, the principal directions are determined by the tensor $\lambda, ij-\lambda, i\lambda_j$. These are the directions in which the osculating geodesic circles of $\overline{G}$ have higher than second order contact with $\overline{G}$. A similar definiton holds for V_n. The hyperosculating directions for G and $\overline{G}$ correspond. If $\overline{V}_n$ is an Einstein space, the hyperosculating directions for G are the Ricci principal directions of V_n. A characterization of the conformal geodesics of a space of constant curvature as well as properties of the conformal geodesics of an Einstein space are given. The mapping of V_n on $\overline{V}_n$ induces a conformal correspondence on the subspaces of each. Simple relationships are found between the conformal geodesics of V_n and of its subspaces. (Received December 27, 1937.)

83. J. W. Green: *The representation of the solution of the generalized Dirichlet problem.*

If T is a bounded domain and $\phi(\gamma)$ a pseudo-continuous function of its boundary elements (see Perkins, Transactions of this Society, vol. 38, no. 1 (1935)), Perkins has shown that there exists a function $u(P)$, harmonic in T, assuming the values $\phi(\gamma)$ continuously at all pseudo-regular boundary elements. The author defines on the boundary elements a positive additive set function $m(\epsilon, P)$, the weak limit of mass distributions on boundaries approximating to that of T. With the aid of this function it is possible to represent $u(P)$ as a Stieltjes integral $\int_\sigma \phi(\gamma) dm(\epsilon_\gamma, P)$. (Received December 21, 1937.)

84. E. V. Huntington: *A rating table for card-matching experiments.*

Suppose all possible runs with a pack of 25 cards composed of 5 suits of 5 cards each are matched against any fixed run taken from a duplicate pack. The first four moments of the distribution of scores are known. (See T. E. Sterne, Science, Nov. 26, 1937; also E. G. Olds, abstract 43-11-428.) Hence the first four moments of the distribution of the average scores obtained in a sequence of n runs can be computed, and a Charlier curve fitted, for each value of n. Hence a "rating table" is computed, giving, for each value of n, the size of the average score which will not be exceeded more than once in so often. For example, in the case of 4 runs, an average score of 7.50 (or 8.25 or 9.25) will not be exceeded more than once in 75 (or 500 or 10,000) times, respectively. In the case of 100 runs, an average score of 5.46 (or 5.60 or 5.77) will not be exceeded more than once in 75 (or 500 or 10,000) times, respectively. The paper will appear in the forthcoming number of the Journal of Parapsychology (Duke University). (Received December 21, 1937.)

85. E. R. Lorch: *On a calculus of operators in reflexive vector spaces.*

A theory of projections is developed in reflexive Banach spaces, $\mathfrak{B}$ (if the adjoint space is denoted by $\overline{\mathfrak{B}}$, the reflexive character is defined by $\overline{\overline{\mathfrak{B}}}=\mathfrak{B}$). It is demonstrated that projection limits of monotone sequences, projection least upper and greatest lower bounds exist for sets of projections satisfying reasonable assumptions. A theory of projection measure, called resolutions of the identity, is developed for certain completely ordered sets. This theory of measure in turn leads to an operational calculus which establishes the existence of a far-reaching homomorphism between

(projection) measurable functions and certain rings of operators in $\mathfrak{B}$. This calculus possesses all the properties attached to that already existing for the special case in which $\mathfrak{B}$ is a Hilbert space, which indicates that the above definition of resolutions of the identity is sufficiently restrictive. (Received December 30, 1937.)

86. E. J. McShane: *Some existence theorems in the calculus of variations. I. Free problems.*

In establishing existence theorems for variation problems requiring $\mathcal{J}(C)$ to be made a minimum, where C is a curve $x=x(t)$ (that is, $x^i=x^i(t)$, $i=1, \cdots, q$) and $\mathcal{J}(C)=\int F(x, x')dt$, the method based on semi-continuity of $\mathcal{J}(C)$ has produced strong results. For isoperimetric problems, however, in which one seeks to minimize $\mathcal{J}(C)$ while holding another integral $G(C)$ constant, that method has been much less fruitful. In this paper an existence theorem for the problem of minimizing $\mathcal{J}(C)$ is established by a method more readily extensible to isoperimetric problems. For each integer n the author finds first a polygon Π_n of not more than n sides which minimizes $\mathcal{J}(C)$ in that class of polygons. The well known properties of minimizing curves for $\mathcal{J}(C)$ (for example, the Weierstrass-Erdmann corner condition) are approximately true for the polygons Π_n. From the restrictions on Π_n thus introduced, one is able, under suitable hypotheses, to show that there is a subsequence of the Π_n which tends both in position and direction to a limit curve C; and it follows at once that $\mathcal{J}(C)=\lim \mathcal{J}(\Pi_n)$, which is the minimum of $\mathcal{J}(C)$. The theorem applies even to certain integrals which are not quasi-regular. (Received December 30, 1937.)

87. F. J. Murray: *Bilinear transformation in Hilbert space.*

A bilinear transformation, $F(f, g)$, is a function of two variables in Hilbert space, linear in f and in g. The elementary theory of linear transformations, with certain modifications, may be extended to $F(f, g)$. For instance, if $F(f, g)$ is continuous at a single point, it is limited, that is, there exists a C such that $\|F(f, g)\| \leq C\|f\| \cdot \|g\|$. Also if $F(f, g)$ is defined for every pair and is closed, it is limited. A bilinear transformation, $F(f, g)$, determines a linear transformation T from $\mathfrak{H}\otimes\mathfrak{H}$ (F. J. Murray and J. von Neumann, Annals of Mathematics, vol. 37, pp. 116–229) to $\mathfrak{H}$; also any such T determines an $F(f, g)$. If T is closable, then the analysis of linear transformations between Hilbert spaces applies (F. J. Murray, Transactions of this Society, vol. 37, pp. 301–338). If $F(f, F(g, h))=F(F(f, g), h)$ and certain other conditions hold, the study of $F(f, g)$ is shown to depend on the theory of rings of operators (cf. first reference above). If one also has $F(f, g)=F(g, f)$, then, for a suitable realization of $\mathfrak{H}$, $F(\phi(\lambda), \psi(\lambda))=\phi(\lambda)\psi(\lambda)$. Some examples are also studied. (Received December 20, 1937.)

88. C. C. Torrance: *An elementary derivation of the Uspensky quadrature formulas.*

In this note the Newton-Cotes and Uspensky quadrature formulas are derived, in a brief and elementary manner, by merely integrating Taylor's formula with a remainder. No reference is made to any interpolation polynomial. By changing constants of integration, a whole new class of quadrature formulas may be obtained. Generalizations of the Bernoulli polynomials are indicated. (Received December 24, 1937.)

89. Audrey Wishard: *Some conditions on functions of bounded types.* Preliminary report.

The background of this paper is the work of Rolf Nevanlinna on functions of bounded type ("beschränktartige Funktionen"). Necessary and sufficient conditions are found that a function, analytic in the right half plane, be of bounded type there. Three types of conditions are found. They are expressed in terms of the behavior of certain integrals along vertical, horizontal, or radial straight lines, rather than along the semi-circles used by Nevanlinna. The properties of the integrals are studied in an effort to determine the asymptotic behavior of the average values of functions of bounded type. (Received December 18, 1937.)

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